Lecture 1.6: Annihilators

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Overview

Last time, we defined the dual of a vector space X to be the set X' of linear scalar functions $X \to K$.

We saw that if $\dim X = n < \infty$, then $X \cong X'$.

Think of $x \in X$ as a column vector, and $\ell \in X'$ as a row vector.

The bilinear scalar product notation

$$(\ell, x) := \ell(x),$$

canonically identifies the double dual X'' with X.

In this lecture, we will study the annihilator of a subspace $Y \leq X$, which is the subspace $Y^\perp \leq X'$ of functions that are zero on all $y \in Y$.

We will determine its dimension (called the codimension of Y), and also understand what $Y^{\perp\perp}$ is.

Annihilators

Definition

Let $Y \leq X$. The set of linear functions that vanish on Y is its annihilator, denoted

$$Y^{\perp} = \{\ell \in X' \mid \ell(y) = 0, \ \forall y \in Y\}.$$

Theorem 1.10

Let $Y \leq X$ with $\dim X < \infty$. Then

$$\dim Y + \dim Y^{\perp} = \dim X.$$

Proof

The annihilator of the annihilator

Definition

The dimension of $Y^\perp$ is called the codimension of Y in X, denoted $\operatorname{codim} Y$.

By Theorem 1.10,

$$\dim Y + \operatorname{codim} Y = \dim X.$$

Since $Y^\perp$ is a subspace of X', its annihilator $Y^{\perp\perp}$ is a subspace of X''.

Theorem 1.11

Assume $\dim X < \infty$ and identify X'' with X. Then $Y^{\perp\perp} = Y$.

Proof

The annihilator of a subset

We can define the annihilator of an arbitrary subset $S \subseteq X$, as

$$S^{\perp} := \{\ell \in X' \mid \ell(s) = 0, \ \forall s \in S\}.$$

Recall that the smallest subspace containing S is

$$\mathrm{Span}(S) = \bigcap_{S \subseteq Y_\alpha \leq X} Y_\alpha.$$

Exercise

Let $S \subseteq X$, and $\dim X < \infty$. Then $S^{\perp} = \mathrm{Span}(S)^{\perp}$.