Lecture 2.7: Change of basis

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Overview

In the previous lecture, we learned how a linear map $T\colon X \to U$ is encoded by a matrix, with respect to an input basis $\mathcal{B}_X$ and output basis $\mathcal{B}_U$.

It is natural to ask how changing the bases changes the matrix.

In this lecture, we will answer this question.

In the special case of $T\colon X \to X$, we will see that two matrices A and B can represent the same linear map if they are similar. That is,

$$A = PBP^{-1}, \qquad \text{for some invertible matrix } P.$$

We will show to how construct such a P, which is called a change of basis matrix.

Change of basis matrices

Let $T\colon X \to U$ be linear, and $x_1, \ldots, x_n$ and $u_1, \ldots, u_m$ be bases.

Since $\dim X = n$ and $\dim U = m$, we have $X \cong K^n$ and $U \cong K^m$. (Let's say $K = \mathbb{R}$.)

An example in $\mathbb{R}^2$

Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be linear, and A the 2×2 matrix w.r.t. the standard basis $e_1, e_2 \in \mathbb{R}^2$.

Let's see what the matrix is with respect to a different basis, $v_1 = \begin{bmatrix} a \\ c \end{bmatrix}$ and $v_2 = \begin{bmatrix} b \\ d \end{bmatrix}$.