Lecture 3.2: Symmetric and skew-symmetric multilinear forms

Matthew Macauley

School of Mathematical & Statistical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 8530, Advanced Linear Algebra

Overview

Loosely speaking, linearity means we can pull apart sums and constants. We have seen:

1. Dual vectors: linear scalar functions $X \to K$
2. Scalar products: bilinear functions $U' \times X \to K$

n. Determinants: functions on n (row or column) vectors where we can break apart certain sums and pull out constants.

These are all examples of multilinear functions.

The determinant is actually a property of a linear map, not a matrix. In this section, we will define and study the determinant in this more abstract context.

The set of k-linear forms $X \times \cdots \times X \to K$ is a vector space of dimension n^k.

The following subclasses of k-linear forms are important subspaces:

- symmetric
- skew-symmetric
- alternating

We will introduce the first two in this lecture.

k-linear forms

Definition

A k-linear form is a function $f \colon X_1 \times \cdots \times X_k \to K$ that is linear in each coordinate.

That is, if we fix $k-1$ inputs, it is linear in the remaining input.

Unless otherwise stated, we will assume that $X := X_1 = \cdots = X_k$.

1. 1-linear forms are linear functions in $X \to K$.
2. 2-linear forms are bilinear forms $X \times X \to K$.
3. A 3-linear form is a function $X \times X \times X \to K$.

The vector space of multilinear forms

Proposition

Let $\dim X = n$. The set of k-linear forms $X \times \cdots \times X \to K$ is a vector space of dimension n^k.

Symmetric and skew-symmetric multilinear forms

Let $f\colon X \times \cdots \times X \to K$ be a k-linear form.

For any permutation $\pi \in S_k$, define the k-linear form πf by

$$(\pi f)(x_1, \dots, x_k) = f\big(x_{\pi_1}, \dots, x_{\pi_k}\big).$$

Definition

A k-linear form is:

1. symmetric if $\pi f = f$ for every permutation $\pi \in S_k$
2. skew-symmetric if $\tau f = -f$ for every transposition $\tau \in S_k$.