Lecture 3.3: Alternating multilinear forms

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Symmetric, skew-symmetric, and alternating forms

Recall that a k-linear form $f\colon X \times \cdots \times X \to K$ is:

- symmetric if $\pi f = f$ for all $\pi \in S_k$,
- skew-symmetric if $\tau f = -f$ for all transpositions $\tau \in S_k$.

Definition

A k-linear form is alternating if $f(x_1, \ldots, x_k) = 0$ whenever $x_i = x_j$ for some $i \neq j$.

It is easy to show that the set of alternating (respectively, symmetric or skew-symmetric) k-linear forms is a subspace of $\mathcal{T}^k(X')$.

Alternating vs. skew-symmetric

Proposition 3.1

Every alternating form is skew-symmetric.

Corollary 3.2

If $1 + 1 \neq 0$, then every skew-symmetric form is alternating.

Alternating forms and linear dependence

Proposition 3.3

If f is alternating and $y_1, \dots, y_k$ is linearly dependent, then $f(y_1, \dots, y_k) = 0$.

Alternating forms and linear independence

Proposition 3.4

If f is alternating and $y_1, \dots, y_n$ is a basis, then $f(y_1, \dots, y_n) \neq 0$.