Lecture 3.4: The determinant of a linear map

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Symmetric, skew-symmetric, and alternating forms

Throughout, $\dim X = n < \infty$. Recall that a k-linear form $f \colon X \times \cdots \times X \to K$ is:

- symmetric if $\pi f = f$ for all $\pi \in S_k$
- skew-symmetric if $\tau f = -f$ for all transpositions $\tau \in S_k$
- alternating if $f(x_1, \ldots, x_k) = 0$ whenever $x_i = x_j$ $(i \neq j)$.

All of these are subspaces of $\mathcal{T}^k(X')$, the space of k-linear forms. *What are their dimensions?*

Goal

Show that the subspace of alternating n-linear forms is 1-dimensional, by verifying

- any two alternating n-linear forms are linearly dependent (see previous lecture)
- there is a non-zero alternating n-linear form.

The determinant of $T \colon \mathbb{R}^n \to \mathbb{R}^n$ is the unique alternating n-linear form satisfying $T(e_1, \ldots, e_n) = 1$.

But we'd still like a definition that doesn't refer to the choice of basis...

The dimension of the subspace of alternating n-linear forms is ≥ 1

Proposition 3.5

There is a nonzero alternating n-linear form.

Determinants, at last

Let $T: X \to X$ be linear. For an alternating n-linear f, define a new alternating n-linear form

$$\bar{T}f: X^n \longrightarrow K, \qquad (\bar{T}f)(x_1, \dots, x_n) = f(Tx_1, \dots, Tx_n).$$

That is, T induces a map $\bar{T}$ on the (1-dimensional) space of alternating n-linear forms:

$$f \longmapsto \bar{T}f.$$

But any linear map on a 1-dimensional space is just scalar multiplication, $x \mapsto \lambda x$. Therefore,

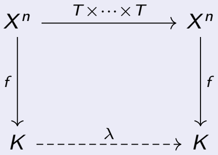
$$\bar{T}: f \longmapsto \lambda f.$$

The scalar λ is called the determinant of T. It satisfies the following.

Universal property of the determinant

Given a linear map $T: X \to X$, there exists a unique scalar λ such that for every alternating n-linear form f,

$$f(Tx_1, \dots, Tx_n) = \lambda f(x_1, \dots, x_n).$$



A few basic properties

If $Tx = cx$, then

$$(\bar{T}f)(x_1, \dots, x_n) = f(Tx_1, \dots, Tx_n) = f(cx_1, \dots, cx_n) = c^n f(x_1, \dots, x_n).$$

Thus, $\det T = c^n$.

It follows that $\det 0 = 0$ and $\det(\mathrm{Id}) = 1$.

Proposition 3.6

For any two linear maps $A, B \colon X \to X$,

$$\det(AB) = (\det A)(\det B).$$

Corollary 3.7

If $A \colon X \to X$ is invertible, then $\det A^{-1} = (\det A)^{-1} \neq 0$.