Lecture 3.6: Minors and cofactors

Matthew Macauley

School of Mathematical & Statistical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 8530, Advanced Linear Algebra

Definitions and motivation

Lemma 3.10

Let $A = [c_1, \ldots, c_n]$ be an $n \times n$ matrix, and define B by adding kc_i to the j^{th} column, for $i \neq j$. Then $\det A = \det B$.

Definition

Let A be an $n \times n$ matrix, and let A_{ij} be the $(n-1) \times (n-1)$ matrix formed by removing the i^{th} row and j^{th} column.

- The (i,j) minor of A is $M_{ij} := \det A_{ij}$.
- The (i,j) cofactor of A is $C_{ij} := (-1)^{i+j} \det A_{ij}$.

Lemma 3.11

Let A be an $n \times n$ matrix with first column e_1, i.e., $A = \begin{bmatrix} 1 & - \\ 0 & A_{11} \end{bmatrix}$. Then $\det A = C_{11}$.

Corollary 3.12

Let A be a matrix whose j^{th} column is e_i. Then

$$\det A = C_{ij}.$$

Laplace expansion

Recall: If the j^{th} column of A is e_i, then $\det A = C_{ij}$.

Theorem (Laplace expansion)

The determinant of A is

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij},$$

for any fixed $j = 1, \dots, n$.

Systems of equations

Consider an invertible matrix, written as an n-tuple of its column vectors:

$$A = (a_1, \dots, a_n) = (Ae_1, \dots, Ae_n).$$

The system of equations $Ax = u$, with $x = \sum_{j=1}^{n} x_j e_j$ can be written

$$\sum_{j=1}^{n} x_j a_j = u.$$

For each k, define the matrix

$$A_k = (a_1, \dots, a_{k-1}, u, a_{k+1}, \dots, a_n),$$

and let's compute its determinant.

A formula for A^{-1}

Theorem (Cramer's rule)

The solution to the system of equations $Ax = u$, with $x = \sum_{j=1}^{n} x_j e_j$ is

$$x_k = \frac{1}{\det A} \sum_{i=1}^{n} C_{ik} u_i.$$

Theorem 3.13

If A is invertible, then the (i, j)-entry of its inverse A^{-1} is

$$(A^{-1})_{ij} = \frac{C_{ji}}{\det A}.$$