Lecture 3.7: Tensors

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What does a tensor product represent?

Consider two vector spaces U, V over K, and say $\dim U = n$ and $\dim V = m$. Then

$$U \cong \{a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \mid a_i \in K\}, \qquad V \cong \{b_{m-1}y^{m-1} + \cdots + b_1 x + b_0 \mid b_i \in K\}.$$

The direct product $U \times V$ has basis

$$\{(x^{n-1}, 0), \dots, (x, 0), (1, 0)\} \cup \{(0, y^{m-1}), \dots, (0, y), (0, 1)\}.$$

An arbitrary element has the form

$$\left(a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,\ b_{m-1}y^{m-1} + \cdots + b_1 y + b_0\right) \in U \times V.$$

Notice that $(3x^i, y^j) \neq (x^i, 3y^j)$ in $U \times V$.

There is another way to "multiply" the vector spaces U and V together. It is easy to check that the following is a vector space:

$$\left\{ \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} c_{ij} x^i y^j \mid c_{ij} \in K \right\}.$$

This is the idea of the tensor product, denoted $U \otimes V$.

Formalizing this is a bit delicate. For example, $3x^i \cdot y^j = x^i \cdot (3y^j) = 3(x^i \cdot y^j)$.

The tensor product in terms of bases

Though we are normally not allowed to "multiply" vectors, we can define it by inventing a special symbol.

Denote the formal "product" of two vectors $u \in U$ and $v \in V$ as $u \otimes v$.

Pick bases $u_1, \dots, u_n$ for U and $v_1, \dots, v_m$ for V.

Definition

The tensor product of U and V is the vector space with basis $\{u_i \otimes v_j\}$.

By definition, every element of $U \otimes V$ can be written uniquely as

$$\sum_{j=1}^{m} \sum_{i=1}^{n} c_{ij}(u_i \otimes v_j).$$

It is immediate that $\dim(U \otimes V) = (\dim U)(\dim V)$.

Remark

Not every multivariate polynomial in x and y factors as a product $p(x)q(y)$.

Not every element in $U \otimes V$ can be written as $u \otimes v$ – called a pure tensor.

A basis-free construction of the tensor product

Given vector spaces U and V, let $F_{U\times V}$ be the vector space with *basis* $U \times V$:

$$F_{U\times V} = \left\{ \sum c_{uv} e_{u,v} \quad \mid \quad u \in U, \;\; v \in V \right\}.$$

For all $u, u' \in U$ and $v, v' \in V$, we "need" the following to hold:

$$e_{u+u',v} = e_{u,v} + e_{u',v} \qquad e_{u,v+v'} = e_{u,v} + e_{u,v'} \qquad e_{cu,v} = ce_{u,v} \qquad e_{u,cv} = ce_{u,v}.$$

Consider the set of "*null sums*" from $F_{U\times V}$:

$$S = \left[\bigcup_{\substack{u,u' \in U \\ v \in V}} e_{u+u',v} - e_{u,v} - e_{u',v} \right] \cup \left[\bigcup_{\substack{u \in U \\ v,v' \in V}} e_{u,v+v'} - e_{u,v} - e_{u,v'} \right]$$
$$\cup \left[\bigcup_{\substack{u\in U, v\in V \\ c\in K}} e_{cu,v} - ce_{u,v} \right] \cup \left[\bigcup_{\substack{u\in U, v\in V \\ c\in K}} e_{u,cv} - ce_{u,v} \right].$$

Let $N_q = \text{Span}(S)$. Denote the equivalence class of $e_{u,v}$ mod N_q as $u \otimes v$.

Definition

The tensor product of U and V is the quotient space $U \otimes V := F_{U\times V}/N_q$.

Why this basis-free construction works

Let W be a vector space with basis $\{w_{ij} \mid 1 \leq i \leq n,\ 1 \leq j \leq m\}$. Define the linear map

$$\alpha \colon W \longrightarrow U \otimes V, \qquad \alpha \colon w_{ij} \longmapsto u_i \otimes v_j.$$

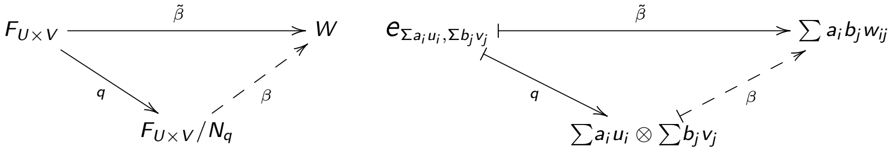
We'd like to define the (inverse) map $\beta \colon U \otimes V \to W$, but to do so, we need a basis for $U \otimes V$. What we *can* do is define a map

$$\tilde{\beta} \colon F_{U \times V} \longrightarrow W, \qquad \tilde{\beta} \colon e_{\Sigma a_i u_i, \Sigma b_j v_j} \longmapsto \sum_{i,j} a_i b_j w_{ij}.$$

Remark (exercise)

The nullspace of $\tilde{\beta}$ contains the nullspace of q.

Since $N_q \subseteq N_{\tilde{\beta}}$, the map $\tilde{\beta}$ *factors through* $F_{U \times V}/N_q := U \otimes V$:



The maps α and β are inverses because $\alpha \circ \beta = \mathrm{Id}_{U \otimes V}$ and $\beta \circ \alpha = \mathrm{Id}_W$.

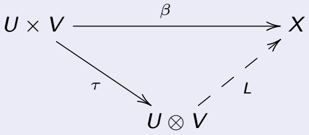
Universal property of the tensor product

Let $\tau\colon U \times V \to U \otimes V$ be the map $(u, v) \mapsto u \otimes v$.

The following says that every bilinear map from $U \times V$ can be "factored through" $U \otimes V$.

Theorem 3.14

For every bilinear $\beta\colon U \times V \to X$, there is a unique linear $L\colon U \otimes V \to X$ such that $\beta = L \circ \tau$.



The universal property can provide us with alternate proofs of some basic results, such as:

(i) $\{u_i \otimes v_j\}$ is linearly independent

(ii) $U \otimes V \cong V \otimes U$

(iii) $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$

(iv) $(U \times V) \otimes W \cong (U \otimes W) \times (V \otimes W)$.

Tensors as linear maps

Proposition 3.15

There is a natural isomorphism

$$U \otimes V \longrightarrow \mathrm{Hom}(U', V), \qquad u \otimes v \longmapsto \big(\ell \mapsto (\ell, u)v\big).$$

The following shows the linear map $\ell \overset{E_{ij}}{\longmapsto} (\ell, u_i)v_j$ in matrix form:

$$\underbrace{\begin{bmatrix} c_1 & \cdots & c_i & \cdots & c_n \end{bmatrix}}_{\ell=\sum c_i \ell_i \in U'} \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & 1 & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{E_{ij} := v_j^T u_i} = \underbrace{\begin{bmatrix} 0 & \cdots & c_i & \cdots & 0 \end{bmatrix}}_{c_i v_j \in V}$$

More generally:

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \otimes \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} = vu^T = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} = \begin{bmatrix} v_1 u_1 & v_1 u_2 & \cdots & v_1 u_n \\ v_2 u_1 & v_2 u_2 & \cdots & v_2 u_n \\ \vdots & \vdots & \ddots & \vdots \\ v_m u_1 & v_m u_2 & \cdots & v_m u_n \end{bmatrix}$$

Tensors as a way to extend an $\mathbb{R}$-vector space to a $\mathbb{C}$-vector space

Let X be an $\mathbb{R}$-vector space with basis $\{x_1, \dots, x_n\}$.

Note that $\mathbb{C}$ is a 2-dimensional $\mathbb{R}$-vector space, with basis $\{1, i\}$.

Suppose $A\colon X \to X$ is a linear map with eigenvalues $\lambda_{1,2} = \pm i$.

If v is an eigenvector v for $\lambda = i$, then $v \notin X$. But v should live in some "extension" of X.

In this bigger vector space, we want to have vectors like

$$zv, \qquad z \in \mathbb{C}, \quad v \in X.$$

What we really want is $\mathbb{C} \otimes X$, which has basis

$$\big\{1 \otimes x_1, \dots, 1 \otimes x_n, i \otimes x_1, \dots, i \otimes x_n\big\} \text{ ``="} \big\{x_1, \dots, x_n, ix_1, \dots, ix_n\big\}.$$

Notice how the associativity that we would expect comes for free with the tensor product, and compare it to the other examples from this lecture:

$$(3i)v = i(3v), \qquad (3x^i)y^j = x^i(3y^j), \qquad (3u)v^T = u(3v^T), \qquad 3u \otimes v = u \otimes 3v.$$