Lecture 4.4: Invariant subspaces

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Invariant subspaces and block diagonal matrices

Throughout, X is an n-dimensional vector space over an algebraically closed field K.

Definition

An invariant subspace of $A\colon X \to X$ is any $Y \leq X$ for which $A(Y) \subseteq Y$.

Suppose $X = Y \oplus Z$, both A-invariant.

If $y_1, \dots, y_k$ and $z_{k+1}, \dots, z_n$ are bases for Y and Z, then the matrix of A with respect to

$$y_1, \dots, y_k, z_{k+1}, \dots, z_n$$

is block-diagonal. It is easy to see how this extends to a sum of A-invariant subspaces,

$$X = Y_1 \oplus \cdots \oplus Y_\ell.$$

Suppose we have a collection $v_1, \dots, v_m$ of generalized eigenvectors:

$$v_{m-1} = (A-\lambda I)v_m, \quad v_{m-2} = (A-\lambda I)^2 v_m, \quad \dots, \quad v_2 = (A-\lambda I)^{m-2} v_m, \quad v_1 = (A-\lambda I)^{m-1} v_m.$$

Notice that $Y = \operatorname{Span}(v_1, \dots, v_m)$ is invariant under both $(A - \lambda I)$ and A.

In this lecture, we will explore what happens when we have multiple genuine eigenvectors, and the invariant subspaces that arise.

An 11×11 example

Suppose $A\colon X \to X$ has characteristic polynomial $p_A(t) = (t-\lambda)^{11}$, and $\dim N_{A-\lambda I} = 4$.

Here is one such possibility for the generalized eigenvectors:

$$v_5 \xmapsto{A-\lambda I} v_4 \xmapsto{A-\lambda I} v_3 \xmapsto{A-\lambda I} v_2 \xmapsto{A-\lambda I} v_1 \xmapsto{A-\lambda I} 0$$

$$w_3 \xmapsto{A-\lambda I} w_2 \xmapsto{A-\lambda I} w_1 \xmapsto{A-\lambda I} 0$$

$$x_2 \xmapsto{A-\lambda I} x_1 \xmapsto{A-\lambda I} 0$$

$$y_1 \xmapsto{A-\lambda I} 0$$

What invariant subspaces do you see?

Let $N_j := N_{(A-\lambda I)^j}$. Notice that

$$\cdots = N_6 = N_5 \supsetneq N_4 \supsetneq N_3 \supsetneq N_2 \supsetneq N_1 \supsetneq 0.$$

The anatomy of an eigenvalue

Key idea

For any $A\colon X \to X$, there is always a basis of generalized eigenvectors of A.

Definition & preview

The algebraic multiplicity of λ is:

- the largest k such that $(t-\lambda)^k$ is a factor of $p_A(t)$
- the maximum number of linearly independent generalized λ-eigenvectors of A
- the number of diagonal entries of λ in the Jordan canonical form.

The geometric multiplicity of λ is:

- $\dim N_{A-\lambda I}$
- the maximum number of linearly independent genuine λ-eigenvectors of A
- the number of Jordan blocks corresponding to λ.

The index of λ is:

- the smallest d such that $N_d = N_{d+1}$
- the "length of the longest chain" of generalized eigenvectors
- the largest m such that $(t-\lambda)^m$ is a factor of $m_A(t)$
- the size of the largest Jordan block corresponding to λ.

A key technical lemma

Lemma 4.7 (HW exercise)

The map $A-\lambda I$ is a well-defined injective map on quotient spaces:

$$A-\lambda I\colon N_{j+1}/N_j \longrightarrow N_j/N_{j-1}, \qquad A-\lambda I\colon \bar{x} \longmapsto \overline{(A-\lambda I)x}.$$

Therefore, $\dim(N_{j+1}/N_j) \leq \dim(N_j/N_{j-1})$.

$$v_5 \xmapsto{A-\lambda I} v_4 \xmapsto{A-\lambda I} v_3 \xmapsto{A-\lambda I} v_2 \xmapsto{A-\lambda I} v_1 \xmapsto{A-\lambda I} 0$$

$$w_3 \xmapsto{A-\lambda I} w_2 \xmapsto{A-\lambda I} w_1 \xmapsto{A-\lambda I} 0$$

$$x_2 \xmapsto{A-\lambda I} x_1 \xmapsto{A-\lambda I} 0$$

$$y_1 \xmapsto{A-\lambda I} 0$$

$$\cdots = N_6 = N_5 \supsetneq N_4 \supsetneq N_3 \supsetneq N_2 \supsetneq N_1 \supsetneq 0.$$