Lecture 4.7: Jordan canonical form

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Overview

Spectral theorem

Let $A \colon X \to X$ be linear. Then

$$X = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k},$$

where $E_{\lambda_j} = \bigcup_{m=1}^{\infty} N_{(A-\lambda_j I)^m}$ is the generalized eigenspace of λ_j.

Moreover, each E_{λ_j} is a direct sum of subspaces invariant under both A and $(A - \lambda_j I)$.

Let's recall an old example where λ has algebraic multiplicity $\dim E_\lambda = 11$ and geometric multiplicity $\dim N_{A-\lambda I} = 4$.

$$v_5 \xmapsto{A-\lambda I} v_4 \xmapsto{A-\lambda I} v_3 \xmapsto{A-\lambda I} v_2 \xmapsto{A-\lambda I} v_1 \xmapsto{A-\lambda I} 0$$

$$w_3 \xmapsto{A-\lambda I} w_2 \xmapsto{A-\lambda I} w_1 \xmapsto{A-\lambda I} 0$$

$$x_2 \xmapsto{A-\lambda I} x_1 \xmapsto{A-\lambda I} 0$$

$$y_1 \xmapsto{A-\lambda I} 0$$

The matrix of A with respect to this is block-diagonal, consisting of Jordan blocks.

Jordan canonical form

A Jordan block is a matrix of the form

$$J_\lambda = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}$$

Every matrix A is similar to a Jordan matrix — a block-diagonal matrix of Jordan blocks:

$$J = \begin{bmatrix} J_{\lambda_1,1} & & & & & & \\ & \ddots & & & & & \\ & & J_{\lambda_1,n_1} & & & & \\ & & & \ddots & & & \\ & & & & J_{\lambda_k,1} & & \\ & & & & & \ddots & \\ & & & & & & J_{\lambda_k,n_k} \end{bmatrix}$$

This is called the Jordan normal form, or Jordan canonical form (JCF) of A.

Summary

Two linear maps $A, B\colon X \to X$ are similar iff they have the same Jordan canonical form.

For each eigenvalue λ, the algebraic multiplicity of λ is the:

- degree of $(t-\lambda)$ in $p_A(t)$
- maximum number of linearly independent generalized λ-eigenvectors of A
- number of diagonal entries of λ in the Jordan canonical form.

The geometric multiplicity of λ is the:

- $\dim N_{A-\lambda I}$
- maximum number of linearly independent genuine λ-eigenvectors of A
- number of Jordan blocks corresponding to λ.

The index of λ is the:

- smallest d such that $N_d = N_{d+1}$ (length of the largest "chain")
- degree of $(t-\lambda)$ in $m_A(t)$
- size of the largest Jordan block corresponding to λ.

A is diagonalizable if:

- X has a basis of genuine eigenvectors
- $m_A(t)$ has no repeated roots
- the Jordan canonical form is a diagonal matrix.

Commuting maps

Lemma 4.12

Let $A, B \colon X \to X$ be commuting linear maps, and $E_\lambda = \bigcup_{j=1}^{\infty} N_{(A-\lambda I)^j}$, the generalized λ-eigenspace of A. Then E_λ is B-invariant.

Theorem 4.13

Let $A, B \colon X \to X$ be commuting linear maps. There is a basis for X consisting of generalized eigenvectors of A and B.

Corollary 4.14

Let $A, B \colon X \to X$ be commuting diagonalizable linear maps. Then they are simultaneously diagonalizable. That is for some invertible $P \colon X \to X$,

$$A = PD_A P^{-1} \qquad \text{and} \qquad B = PD_B P^{-1}.$$