Lecture 4.8: Generalized eigenfunctions of differential operators

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Motivation: ODEs with repeated roots

Recall how to solve the differential equation $y'' - 3y' + 2y = 0$:

- Look for a solution of the form $y(t) = e^{rt}$.
- Plug back in to get $e^{rt}(r^2 - 3r + 2) = 0$, and so $r = 1$ or $r = 2$.
- The general solution is thus $y(t) = C_1 e^t + C_2 e^{2t}$.

A "problem case" occurs when the "characteristic equation" has repeated roots.

For example, consider $y'' - 2\lambda y' + \lambda^2 y = 0$.

The same process gives $r_1 = r_2 = \lambda$, so we only get one solution, $y_1(t) = e^{\lambda t}$.

However, the solution space is two-dimensional. It turns out that $y_2(t) = te^{\lambda t}$ is also a solution.

In this lecture, we'll see how this arises as a generalized eigenfunction of a differential operator.

The derivative operator

Clearly, $y_1(t) = e^{\lambda t}$ is an eigenfunction of $D = \frac{d}{dt}$.

Equivalently, it is in $N_{D-\lambda I}$, and solves the ODE

$$(D - \lambda I)y = 0 \qquad \Leftrightarrow \qquad \left(\tfrac{d}{dt} - \lambda\right)y = 0 \qquad \Leftrightarrow \qquad y' - \lambda y = 0.$$

Generalized eigenfunctions in $N_{(D-\lambda I)^2}$ are solutions to the second order ODE

$$(D - \lambda I)^2 y = 0, \qquad \Leftrightarrow \qquad \left(\tfrac{d}{dt} - \lambda\right)^2 y = 0, \qquad \Leftrightarrow \qquad y'' - 2\lambda y' + \lambda^2 y = 0$$

It is easy to see that $y_2(t) = te^{\lambda t}$ is in $N_{(D-\lambda I)^2}$, because

$$D(y_2) = D\big(te^{\lambda t}\big) = e^{\lambda t} + \lambda te^{\lambda t} = y_1 + \lambda y_2.$$

Similarly, $y_3(t) = \frac{1}{2!}t^2e^{\lambda t}$ is in $N_{(D-\lambda I)^3}$, because

$$D(y_3) = D\big(\tfrac{1}{2!}t^2e^{\lambda t}\big) = te^{\lambda t} + \lambda\tfrac{1}{2!}t^2e^{\lambda t} = y_2 + \lambda y_3.$$

Repeating in this manner, we see that the generalized eigevectors for D are:

$$\cdots \xmapsto{D-\lambda I} \tfrac{1}{4!}t^4e^{\lambda t} \xmapsto{D-\lambda I} \tfrac{1}{3!}t^3e^{\lambda t} \xmapsto{D-\lambda I} \tfrac{1}{2!}t^2e^{\lambda t} \xmapsto{D-\lambda I} te^{\lambda t} \xmapsto{D-\lambda I} e^{\lambda t} \xmapsto{D-\lambda I} 0$$

The generalized eigenspace of D for eigenvalue λ is thus

$$E_\lambda = \big\{p(t)e^{\lambda t} \mid p \in K[t]\big\}.$$

Systems of linear differential equations

Consider the linear system $x' = Ax$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

It is easy to check that if $Av = \lambda v$, then $x(t) = e^{\lambda t} v$ is a solution.

Thus, the general solution is

$$x(t) = C_1 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} C_1 e^{3t} + C_2 e^{-t} \\ 2C_1 e^{3t} - 2C_2 e^{-t} \end{bmatrix}.$$

Now, consider an example that has only one eigenvector:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad x_1(t) = e^{\lambda t} v_1 = e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

In an ODE course, one is taught to look for a solution of the form

$$x_2(t) = te^{-2t} v + e^{-2t} w,$$

and solve for v and w.

We'll see that what we're really doing is finding generalized eigenvectors of A.

Solving $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with repeated eigenvalues

Suppose that $Av = \lambda v$, and so $x_1(t) = e^{\lambda t} v$ is a solution. Consider

$$x_2(t) = te^{\lambda t} v + e^{\lambda t} w,$$

and plug this back into $x' = Ax$:

- $Ax_2 = te^{\lambda t} Av + e^{\lambda t} Aw$.
- $x_2' = (e^{\lambda t} v + \lambda te^{\lambda t} v) + \lambda e^{\lambda t} w$.

Equate like terms and divide by $e^{\lambda t}$:

- $te^{\lambda t}$: $Av = \lambda v$
- $e^{\lambda t}$: $Aw = v + \lambda w$.

In other words, $v = v_1$ is the eigenvector, and $w = v_2$ a generalized eigenvector. The general solution is

$$x(t) = C_1 x_1(t) + C_2 x_2(t) = C_1 e^{\lambda t} v_1 + C_2 e^{\lambda t}(tv_1 + v_2).$$

In summary, if the generalized eigenvectors of A are

$$v_2 \xmapsto{A-\lambda I} v_1 \xmapsto{A-\lambda I} 0$$

then the generalized eigenvectors of $A - \frac{d}{dt}$ are

$$\cdots \xmapsto{A-\frac{d}{dt}} e^{\lambda t}\left(\tfrac{t^2}{2!} v_1 + tv_2 + v_3\right) \xmapsto{A-\frac{d}{dt}} e^{\lambda t}(tv_1 + v_2) \xmapsto{A-\frac{d}{dt}} e^{\lambda t} v_1 \xmapsto{A-\frac{d}{dt}} 0$$

A Jordan matrix perspective

Formally, suppose we have the system $x' = Ax$, and $A = PJP^{-1}$.

$$(P^{-1}x)' = J(P^{-1}x), \qquad \text{let } z = P^{-1}x \quad \Leftrightarrow \quad x = Pz.$$

Now, we just have to analyze $z' = Jz$ for a Jordan matrix.

The solution is

$$z = e^{\lambda t} \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & \cdots & \frac{t^{k-1}}{(k-1)!} \\ & 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{k-2}}{(k-2)!} \\ & & 1 & t & \cdots & \frac{t^{k-3}}{(k-3)!} \\ & & & \ddots & \ddots & \vdots \\ & & & & 1 & t \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_{n-1} \\ C_n \end{bmatrix} = e^{Jt}c.$$

It is easy to extend this to one where J has multiple Jordan blocks.

An example

Let's return to our example of $x' = Ax$, with only one eigenvector:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad x_1(t) = e^{\lambda t} v_1 = e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The Jordan canonical form $A = PJP^{-1}$ is

$$\begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

The solution is $x = Pz$, where $z = e^{\lambda t} e^{Jt} c$:

$$x(t) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} e^{-2t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = e^{-2t} \begin{bmatrix} 1 & t+1 \\ 1 & t \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} C_1 e^{-2t} + C_2 e^{-2t}(t+1) \\ C_1 e^{-2t} + C_2 t e^{-2t} \end{bmatrix}.$$

Notice that we can rearrange terms to get this into a familiar form:

$$x(t) = C_1 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-2t} \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = C_1 e^{-2t} v_1 + C_2 e^{-2t} (t v_1 + v_2).$$

In other words, the generalized eigenvectors are:

$$e^{-2t} \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \xmapsto{A - \frac{d}{dt}} e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xmapsto{A - \frac{d}{dt}} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$