Lecture 5.2: Orthogonality

Matthew Macauley

School of Mathematical & Statistical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 8530, Advanced Linear Algebra

Overview

Last time, we defined an inner product space, as a vector space with a symmetric positive-definite bilinear form.

This generalized the notion of the dot product in standard Euclidean space, $\mathbb{R}^n$.

The Cauchy-Schwarz and triangle inequalities allowed us to define analogues of

- *length*: $||x|| = \sqrt{\langle x, x\rangle}$
- *angle*: $\cos\theta = \dfrac{\langle x, y\rangle}{||x||\,||y||}$.

If an inner product space is a generalization of Euclidean space, then orthogonal is the analogue of perpendicular.

Definition

Two vectors $x, y \in X$ are orthogonal if $\langle x, y\rangle = 0$. We write $x \perp y$.

Pythagorean theorem

If $x \perp y$, then $||x+y||^2 = ||x||^2 + ||y||^2$.

Why orthogonal bases are nice

Let $x_1, \dots, x_n$ be an orthogonal basis (not necessarily orthonormal).

Given $v \in X$, we can write

$$v = a_1 x_1 + \cdots + a_n x_n.$$

We can find a formula for a_i by applying the linear map $\langle -, x_i \rangle$ to both sides:

$$a_i = \frac{\langle v, x_i \rangle}{\langle x_i, x_i \rangle}.$$

Remark

We can project x onto a vector $u \in X$ by defining

$$\operatorname{proj}_u x = \frac{\langle x, u \rangle}{\langle u, u \rangle}, \qquad \operatorname{Proj}_u x = \frac{\langle x, u \rangle}{\langle u, u \rangle} u.$$

Definition

The vectors $x_1, \dots, x_k$ in X is orthonormal if

$$\langle x_i, x_j \rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

Orthonormal bases

Key idea

- Orthogonal is the abstract version of "*perpendicular*."
- Orthonormal means "*perpendicular and unit length*."

Orthonormal bases are really desirable!

If $x_1, \dots, x_n$ is an orthonormal basis, $x = \sum_{i=1}^{n} a_i x_i$, and $y = \sum_{i=1}^{n} b_i x_i$, then

- $a_i = \text{proj}_{x_i} x = \langle x, x_i \rangle$
- $\langle x, y \rangle = \sum_{i=1}^{n} a_i b_i$
- $||x||^2 = \sum_{i=1}^{n} a_i^2$.

Remark

If the columns of a matrix A are orthonormal, then $A^T A = I$.

Examples of orthogonality

Let's compare what orthogonality means in several inner product spaces:

1. $X = \mathbb{R}^n$, with the standard dot product.
2. $X = \mathbb{R}^2$, with inner product

$$\left\langle a_1 e_1 + a_2 e_2, b_1 e_1 + b_2 e_2 \right\rangle = \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 2a_1 b_1 + a_1 b_2 + b_1 a_2 + 2a_2 b_2.$$

Next, for fun, we'll do a quick high-level tour of how orthogonality arises in differential equations, involving:

1. Fourier series
2. Sturm-Liouville theory

Fourier series

Consider the space $X = \text{Per}_{2\pi}(\mathbb{R})$ of 2π-periodic piecewise functions, with the inner product

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

The set

$$\left\{\tfrac{1}{\sqrt{2}},\ \cos x,\ \cos 2x, \dots\right\} \cup \left\{\sin x,\ \sin 2x,\ \dots\right\}.$$

is an orthonormal basis w.r.t. to this inner product.

Thus, we can write each $f(x) \in \text{Per}_{2\pi}$ *uniquely* as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$

where

$$a_n = \text{proj}_{\cos nx}(f) = \langle f, \cos nx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \text{proj}_{\sin nx}(f) = \langle f, \sin nx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Remark

There are technical details that need to be addressed regarding infinite sums and convergence, but those are beyond the scope of this class.

Legendre polynomials

The following is an eigenvalue problem $Ly = \lambda y$, on $(-1,1)$:

$$-\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}y\right] = \lambda y.$$

The eigenvalues are $\lambda_n = n(n+1)$, $n \in \mathbb{N}$, and the eigenfunctions solve Legendre's equation:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0.$$

For each n, one solution is a degree-n "Legendre polynomial"

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right].$$

They are orthogonal with respect to the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$.

It can be checked that

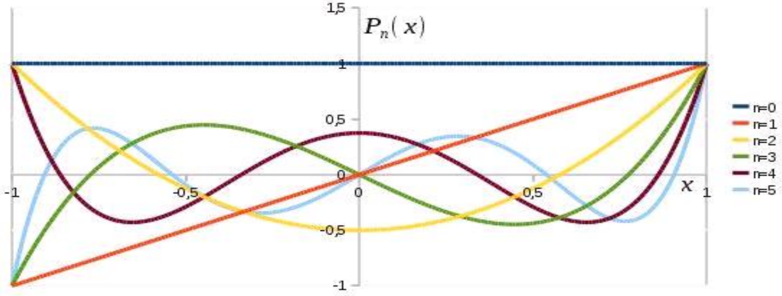
$$\langle P_m, P_n\rangle = \int_{-1}^{1} P_m(x)P_n(x)\,dx = \frac{2}{2n+1}\delta_{mn}.$$

By orthogonality, every function f, continuous on $-1 < x < 1$, can be expressed using Legendre polynomials:

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x), \qquad \text{where} \quad c_n = \frac{\langle f,\, P_n\rangle}{\langle P_n,\, P_n\rangle} = (n+\tfrac{1}{2})\,\langle f,\, P_n\rangle.$$

Legendre polynomials

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \tfrac{1}{2}(3x^2 - 1)$$
$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x)$$
$$P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3)$$
$$P_5(x) = \tfrac{1}{8}(63x^5 - 70x^3 + 15x)$$
$$P_6(x) = \tfrac{1}{8}(231x^6 - 315x^4 + 105x^2 - 5)$$
$$P_7(x) = \tfrac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)$$



Chebyshev polynomials

The following is a "weighted" eigenvalue problem $Ly = \lambda w(x) y$ on $[-1,1]$:

$$-\frac{d}{dx}\left[\sqrt{1-x^2}\,\frac{d}{dx}y\right] = \lambda\frac{1}{\sqrt{1-x^2}}y.$$

The eigenvalues are $\lambda_n = n^2$ for $n \in \mathbb{N}$, and the eigenfunctions solve Chebyshev's equation:

$$(1-x^2)y'' - xy' + n^2 y = 0.$$

For each n, one solution is a degree-n "Chebyshev polynomial," defined recursively by

$$T_0(x) = 1, \qquad T_1(x) = x, \qquad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

They are orthogonal with respect to the inner product $\langle f, g\rangle = \displaystyle\int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}\,dx$.

It can be checked that

$$\langle T_m, T_n\rangle = \int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}\,dx = \begin{cases} \frac{1}{2}\pi\delta_{mn} & m \neq 0, n \neq 0 \\ \pi & m = n = 0 \end{cases}$$

By orthogonality, every function $f(x)$, continuous for $-1 < x < 1$, can be expressed using Chebyshev polynomials:

$$f(x) \sim \sum_{n=0}^{\infty} c_n T_n(x), \qquad \text{where} \quad c_n = \frac{\langle f,\ T_n\rangle}{\langle T_n,\ T_n\rangle} = \frac{2}{\pi}\langle f,\ T_n\rangle, \ \text{ if } n > 0.$$

Chebyshev polynomials (of the first kind)

$$T_0(x) = 1 \qquad T_4(x) = 8x^4 - 8x^2 + 1$$
$$T_1(x) = x \qquad T_5(x) = 16x^5 - 20x^3 + 5x$$
$$T_2(x) = 2x^2 - 1 \qquad T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$
$$T_3(x) = 4x^3 - 3x \qquad T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$$