Lecture 6.2: Spectral resolutions

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Eigenvalues and eigenvectors of self-adjoint maps

Theorem 6.1

A self-adjoint linear map $H\colon X \to X$ has only real eigenvalues, and a set of eigenvectors that forms an orthonormal basis of X.

Proof

We will show that:

1. H has only real eigenvalues
2. H has no (purely) generalized eigenvectors
3. eigenvectors corresponding to different eigenvalues are orthogonal.

Unitary diagonalization

Theorem 6.1

A self-adjoint linear map $H \colon X \to X$ has only real eigenvalues, and a set of eigenvectors that forms an orthonormal basis of X.

Corollary 6.2

If $H \colon X \to X$ is self-adjoint, then H is diagonalizable by a unitary matrix U. That is,

$$H = UDU^*, \qquad \text{where } U^*U = I.$$

Orthogonal projections onto eigenspaces

If $H\colon X \to X$ is self-adjoint with distinct eigenvectors $\lambda_1, \dots, \lambda_k$, then we can write

$$X = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}, \qquad \text{where } E_{\lambda_j} = N_{A-\lambda_j I},$$

i.e., E_{λ_j} is the eigenspace for λ_j.

This means we can write any $x \in X$ as

$$x = x^{(1)} + \cdots + x^{(k)}, \qquad \text{where } x^{(j)} \in E_{\lambda_j}.$$

Note that

$$Hx = \lambda_1 x^{(1)} + \cdots + \lambda_k x^{(k)}.$$

Denote the projection of $x \in X$ onto the eigenspace E_{λ_j} by

$$P_j\colon X \longrightarrow X, \qquad P_j\colon x \longmapsto x^{(j)}.$$

Remark

The orthogonal projection maps satisfy

(i) $P_i P_j = 0$ if $i \neq j$

(ii) $P_i^2 = P_i$

(iii) $P_i^* = P_i$.

Spectral resolutions

Definition

The decompositions

$$I = \sum_{j=1}^{k} P_j, \qquad H = \sum_{j=1}^{k} \lambda_j P_j$$

are called a resolution of the identity, and the spectral resolution of H, respectively.

Corollary 6.2 (self-adjoint maps are unitarily diagonalizable) can now be re-stated as:

Theorem 6.3

If $H\colon X \to X$ is self-adjoint, then there is a resolution of the identity, and a spectral resolution of H.

Functions of self-adjoint maps

Key idea

Spectral resolutions allow us to define functions on a self-adjoint map.

For example if $H\colon X \to X$ is self-adjoint with spectral resolution $H = \sum_{j=1}^{k} \lambda_j P_j$, then

- $H^2 = \sum_{j=1}^{k} \lambda_j^2 P_j$
- $H^m = \sum_{j=1}^{k} \lambda_j^m P_j$
- $p(H) = \sum_{j=1}^{k} p(\lambda_j) P_j$, for any polynomial $p(t)$
- $e^H = \sum_{j=1}^{k} e^{\lambda_j} P_j$
- $f(H) = \sum_{j=1}^{k} f(\lambda_j) P_j$, for any function $f(t)$ defined on $\lambda_1, \dots, \lambda_k$.