# Math 4120, Midterm 1. October 12, 2022

Write your answers for these problems *directly on this paper*. You should be able to fit them in the space given.

1. (10 points) Let $G$ be a group with an index-5 subgroup $H = \langle a, b \rangle$, and an index-4 subgroup $K$. *Be as specific as possible with your answers.*

   (a) There are ______________________ left cosets of $K$, and ______________________ right cosets.

   (b) If $c \in H$, then $\langle a, b, c \rangle =$ ____________________. Otherwise, $\langle a, b, c \rangle =$ ____________________.

   (c) If $HK$ is a subgroup, then it must be ________________________________________________.

   (d) If $K \lneq N \lneq G$, then $N \trianglelefteq G$ because ________________________________________________.

   (e) The normalizer of $K$ can have index $[G : N_G(K)] =$ ____________________ (list all possibilities).

   (f) The possible size(s) of the conjugacy class $\mathrm{cl}_G(K)$ is/are ________________ (list all possibilities).

   (g) If $H$ is *not* normal, then the possible size(s) of its conjugacy class $\mathrm{cl}_G(H)$ is/are ______________.

2. (6 points) Finish the following definition. Don't just list the properties; actually define what they mean, correctly using "there exists" and "for all" (i.e., $\exists$ and $\forall$), where appropriate.

   - A *group* is a set $G$ with a binary operation $*$ satisfying the following properties:

3. (6 pts) Give 3 different equivalent conditions of what it means for a subgroup $H \leq G$ to be *normal* in $G$: one involving *cosets*, one involving *conjugate subgroups*, and one involving *conjugates of elements*. As before, make sure that you use "for all" or $\forall$, where appropriate.

   (i)

   (ii)

   (iii)

4. (8 pts) Let $H = \{1, h\} \leq G$ be a subgroup of order 2. Prove that if $H \trianglelefteq G$, then $H$ is contained in the *center*, $Z(G) := \big\{ z \in G \mid zg = gz,\ \forall g \in G \big\}$.

5. (15 points) Consider the *dihedral group* $D_3 = \langle r, f \mid r^3 = f^2 = 1,\ rf = fr^2 \rangle$.

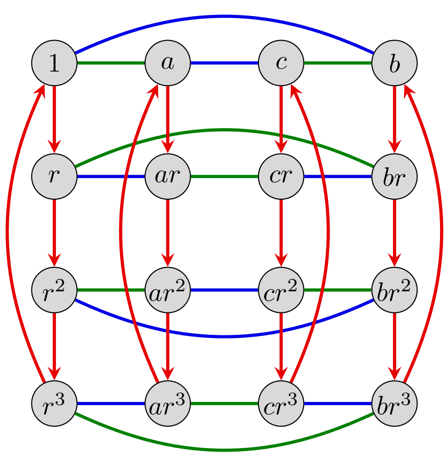
   (a) Draw a *Cayley graph* of $D_3$, and label the nodes with elements.

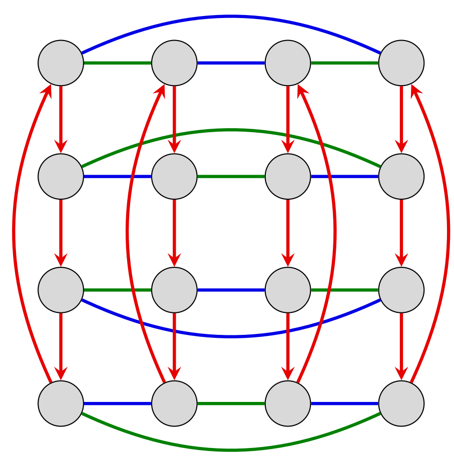
   (b) Draw the *subgroup lattice*, listing subgroups by generator(s). Label each edge between $K \leq H$ with the corresponding index, $[H : K]$. Then, use circles to partition the subgroups into conjugacy classes.

   (c) Draw the *cycle graph* of $D_3$.

*Cayley graph* *subgroup lattice* *cycle graph*

6. (24 pts) Consider the group $G$ whose Cayley graph is shown below, with subgroups $H = \langle r \rangle$ and $K = \langle a, b \rangle$.



   (a) Inside each node on the right-most diagram, write the *order* of the corresponding element.

   (b) These subgroups are isomorphic to familiar groups: $H \cong$ ____________ and $K \cong$ ____________.

   (c) Find the left cosets of $H = \langle r \rangle$, then find the right cosets. Write them as subsets, or describe them in words (e.g., "the rows" or "the columns").

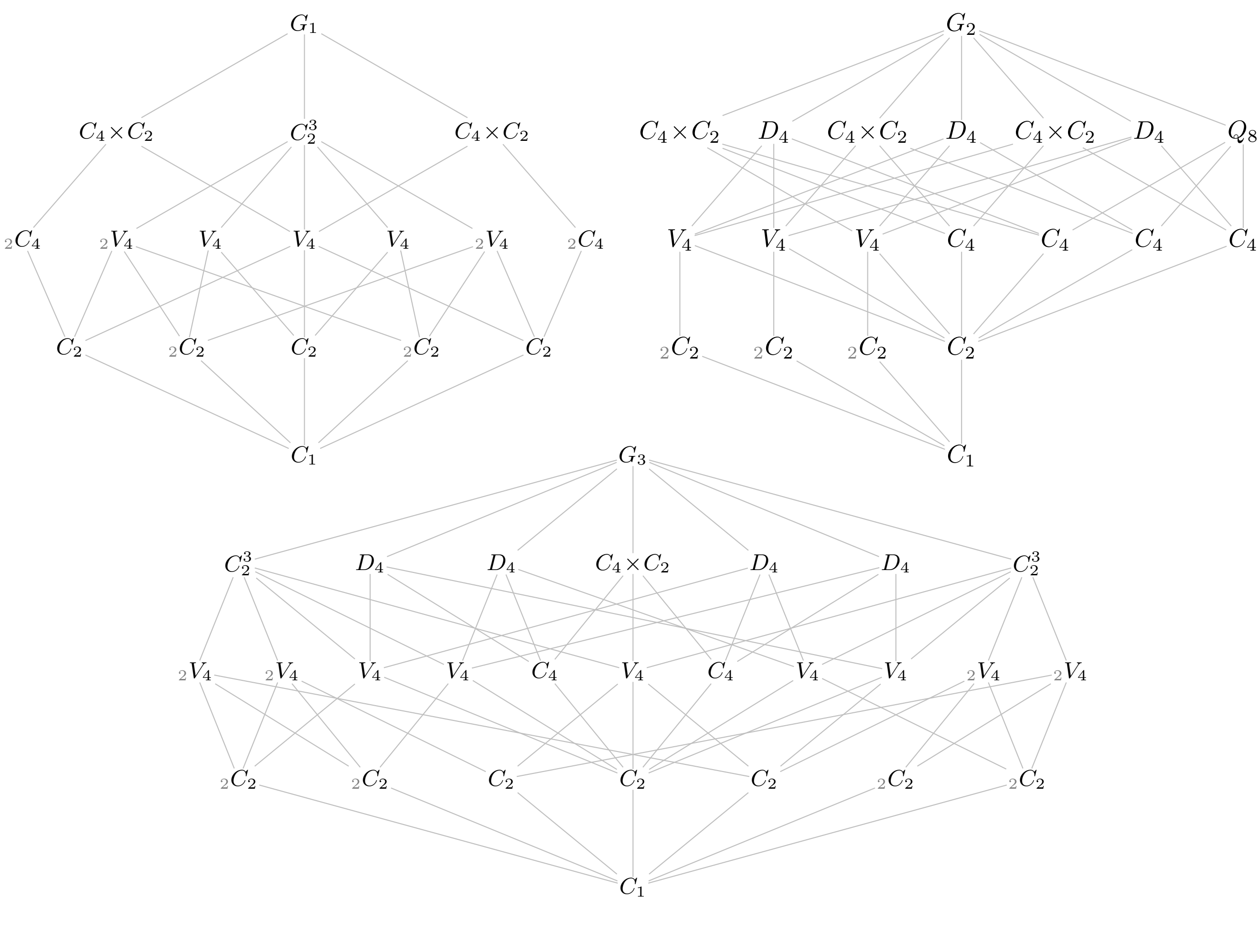
   (d) Find the left cosets of $K = \langle a, b \rangle$, then find the right cosets. Write them as subsets, or describe them in words (e.g., "the rows" or "the columns").

(e) Find the normalizers of $H$ and $K$. Write them by generator(s), and say what familiar group each is isomorphic to.

(f) Find all conjugate subgroups to $H$ and to $K$. Write each group by generator(s).

(g) The center of this group is $Z(G) =$ ________________________________________.

(h) Subgroup lattices of three groups of order 16 are shown below. Recall that the nodes are "collpased" to represent conjugacy classes, and those of the non-normal subgroups have a "left subscript" that denotes their size. Determine which is the actual subgroup lattice of $G$, and fully justify your answer. [*Hint*: One way to do this is to eliminate the two incorrect lattices, and you will get points for each one that is done successfully – find a reason why it cannot be $G$.]



7. (10 points) Make a list of all abelian groups of order $216 = 2^3 \cdot 3^3$. That is, every abelian group of order 216 should be isomorphic to precisely <u>one</u> group on your list. Feel free to write e.g., $C_2^2 := C_2 \times C_2$ for short. Finally, circle every group on your list that is *cyclic*.

8. (9 points) Given a subgroup $H \leq G$ and element $x \in G$, prove that $xHx^{-1} = \{xhx^{-1} \mid h \in H\}$ is a subgroup of $G$.

9. (12 points) Answer the following about permutations and the symmetric group $S_n$.

   (a) The *order* of the permutation $\pi = (123)(456789)$ in $S_9$ is ____________________

   (b) The *inverse* of the permutation $\pi = (123)(456789)$ in $S_9$ is $\pi^{-1} =$ ____________________

   (c) The product of $\pi = (12)(465)$ and $\sigma = (1423)(56)$ in $S_6$, is $\pi\sigma =$ ____________________.

   (d) The subgroup $\langle (12), (45) \rangle$ of $S_5$ is isomorphic to ____________________.

   (e) The *conjugacy class* of $\pi = (12)$ in $S_3$ is $\mathrm{cl}_{S_3}(\pi) =$ ____________________.

   (f) The alternating group $A_n$ consists of all __________ permutations, and has order __________.