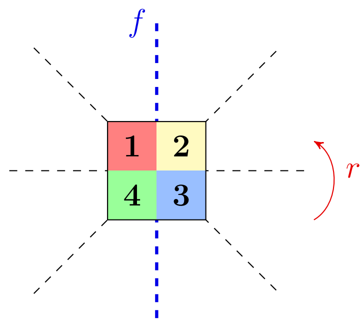
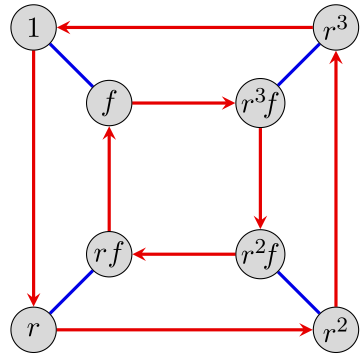
1. The eight symmetries of a square form a group that we will call **Sq**, generated by a 90° counterclockwise rotation r, and a horizontal flip f. A Cayley diagram is shown below.





   (a) For each axis of reflection, express the symmetry across it in terms of r and f.

   (b) Find all *minimal* generating sets. [*Hint*: There are 12.]

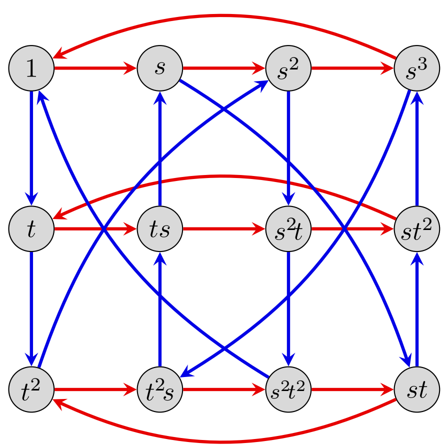
   (c) Let $s = f$ and $t = r^3 f = fr$. Draw a Cayley diagram using s and t as generators.

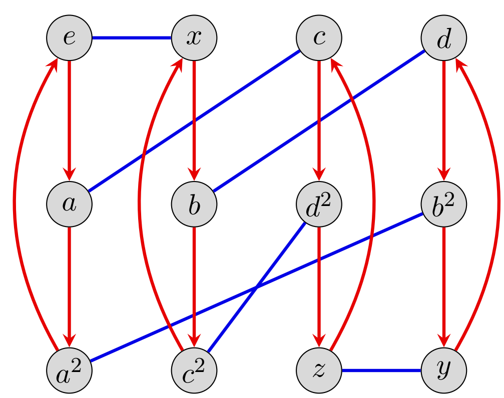
   (d) Write a presentation of the form $\mathbf{Sq} = \langle r, f \mid \cdots \rangle$.

   (e) Write a presentation of the form $\mathbf{Sq} = \langle s, t \mid \cdots \rangle$.

   (f) Construct a *Cayley table* for this group, ordered $1, r, r^2, r^3, f, rf, r^2f, r^3f$. Describe how the rotations and reflections are "clustered" in this table.

2. The Cayley diagrams of two groups of size 12 are shown below.





   (a) Create a Cayley table for each group. (For consistency, please order the elements in the first group by $1, t^2, s^2t, t, s^2, s^2t^2, s, st^2, t^2s, st, s^3, ts$, and those in second by $e, x, y, z, a, b, c, d, a^2, b^2, c^2, d^2$.)

   (b) Find the inverse of each element.

   (c) Find the *order* of each g: the minimal $k > 0$ such that $g^k = e$, denoted $|g|$.

   (d) Write a presentation for each group.

   (e) Determine whether or not these two groups are isomorphic. Justify your answer.

   (f) Squint your eyes. Do you see any patterns in these tables?

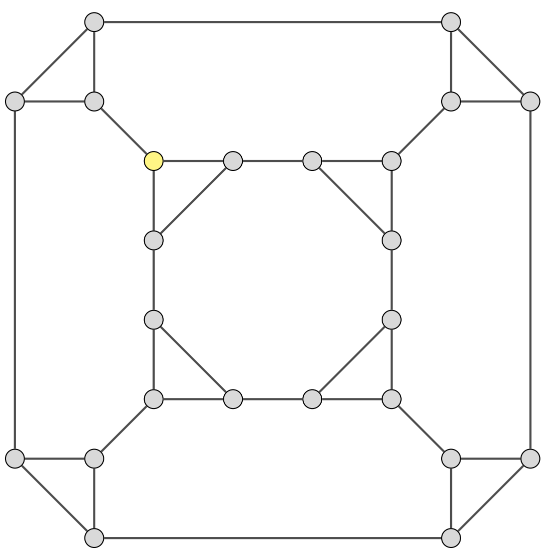
3. In this problem, we will define two variations of the $\mathbf{Coin}_2$ group from lecture. We will consider two types of tiles, and declare the following to be the "*home state*" of each:

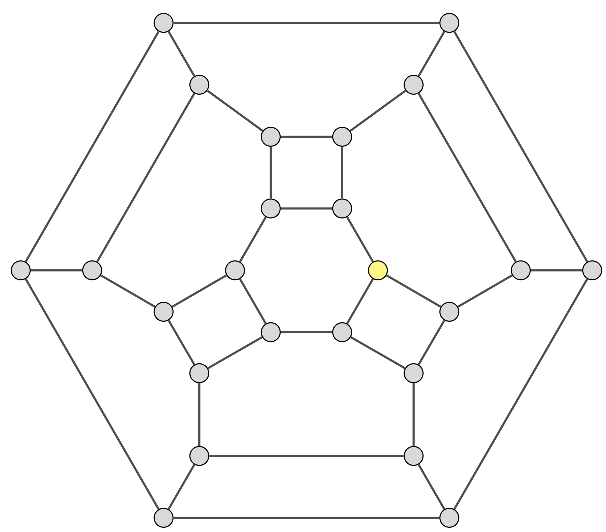
Our first group is $\mathbf{Coin}_3 = \langle c, t \rangle$, where c "*cyclicaly shifts*" the entries, and t "*toggles*" the color of the leftmost square:

Our second group is $\mathbf{Box}_2 = \langle r, s \rangle$, where r "*rotates*" the squares counterclockwise, and s "*swaps*" the squares on the top row.

Note that the square tiles don't actually need to be shaded. An alternate way to denote the colors of the 3×1 dominos is to underline any number with a black background. For example, using this convention, the "home state" would be written $\underline{\mathbf{1}}\,\mathbf{2}\,\underline{\mathbf{3}}$.

(a) Both of these groups have 24 actions. Draw a Cayley diagram for each, with the nodes labeled by configurations. It is helpful to know that the one for $\mathbf{Coin}_3$ can be arranged on a *truncated cube*, whose skeleton is shown below (left). A Cayley diagram for $\mathbf{Box}_2$ can be arranged on a *truncated octahedron*, shown below (right). But the "home state" at the yellow node.



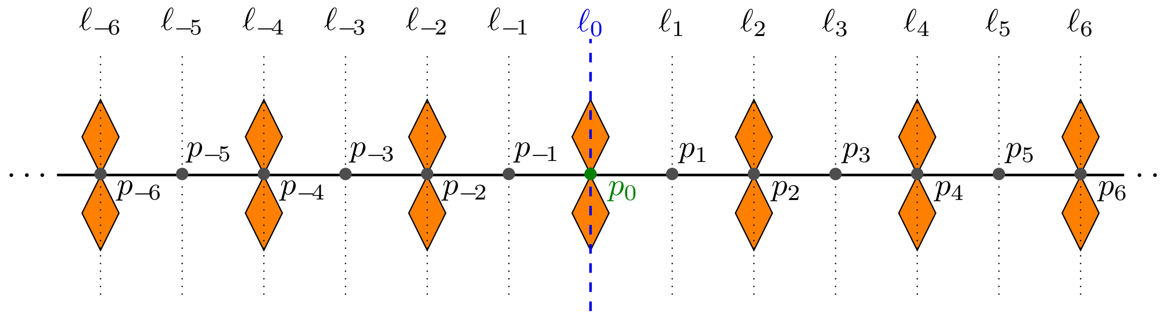


(b) On a fresh copy of these graphs, color the edges of the Cayley graph and label each node by its *order*.

(c) Write down a presentation for each of these groups.

(d) Are these groups isomorphic? Justify your answer.

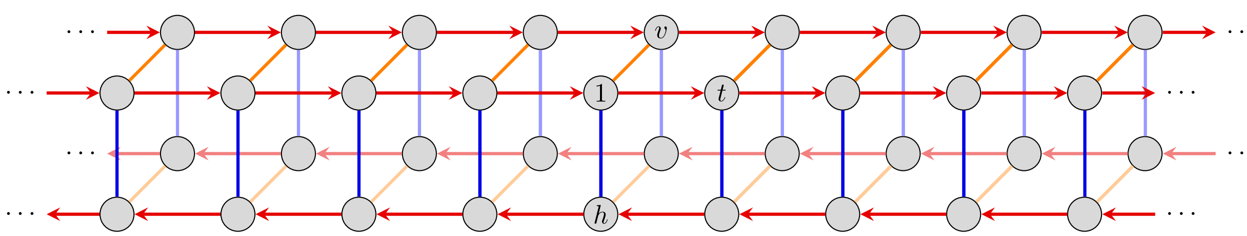
4. Consider the frieze shown below:



Let t be a minimal translation to the right, h_i a reflection across ℓ_i, and r_j a 180° rotation around p_j. Let v be the vertical reflection and $g_i = t^i v$ a glide reflection. A presentation for the frieze group is

$$\mathbf{Frz}_1 := \langle t, h, v \mid v^2 = h^2 = 1,\, th = ht^{-1},\, tv = vt,\, hv = vh \rangle,$$

where $h = h_0$. A Cayley diagram is shown below.

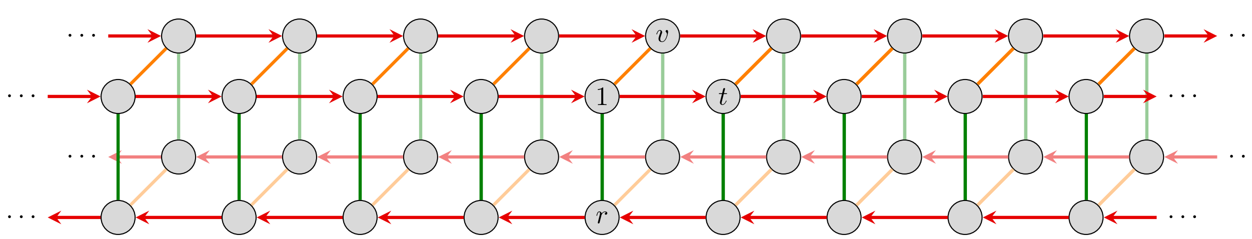


(a) Every symmetry is either a translation t^i, glide reflection g_j, rotation r_k, horizontal reflection h_ℓ, or the vertical reflection v. Label the vertices of this Cayley diagram with elements written in this form.

(b) Now, repeat the previous part, but with for the Cayley diagram for the presentation

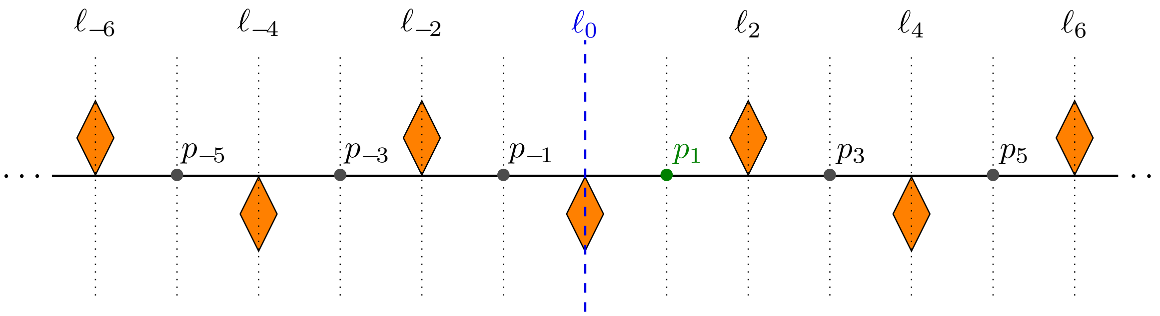
$$\mathbf{Frz}_1 := \langle t, r, v \mid v^2 = r^2 = 1,\, tr = rt^{-1},\, tv = vt,\, rv = vr \rangle,$$

where $r = r_0$. A Cayley diagram is shown below.



(c) Determine which symmetries $t^i h t^{-i}$ and $t^i r t^{-i}$ are for each $i \in \mathbb{Z}$.

5. The subgroup $\mathbf{Frz}_2 = \langle g, h \rangle$ of the frieze group from the previous problem, where $g = g_1 = tv$, is the symmetry group of the following frieze:

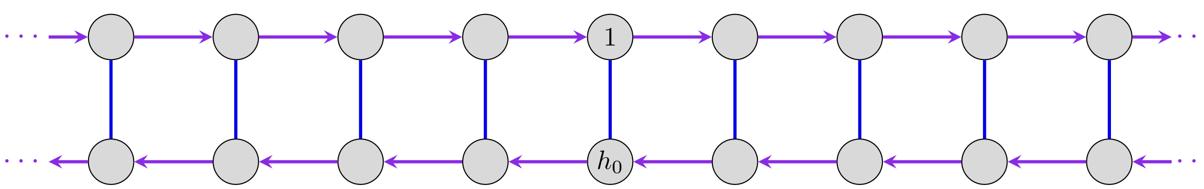


Two presentations for this frieze group are

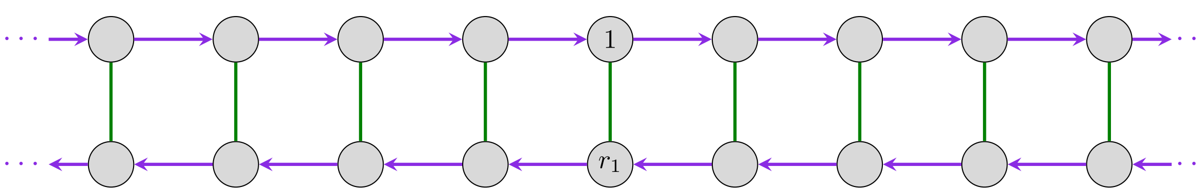
$$\mathbf{Frz}_2 = \langle g, h \mid h^2 = 1,\ gh = hg^{-1} \rangle = \langle g, r \mid r^2 = 1,\ gr = rg^{-1} \rangle,$$

where $h = h_0$ and $r = r_1$.

(a) Label the following Cayley diagram for $\mathbf{Frz}_2 = \langle g, h \rangle$ with elements of the form g^i, h_j, and r_k for $i, j, k \in \mathbb{Z}$.



(b) Repeat the previous part for the following Cayley diagram:



(c) Draw a Cayley diagram for $\mathbf{Frz}_2 = \langle g, h, r \rangle$ and label the nodes with actions of the form g^i, h_j, and r_k, for $i, j, k \in \mathbb{Z}$.

(d) Determine which symmetries $g^i h g^{-i}$ and $g^i r g^{-i}$ are for each $i \in \mathbb{Z}$.