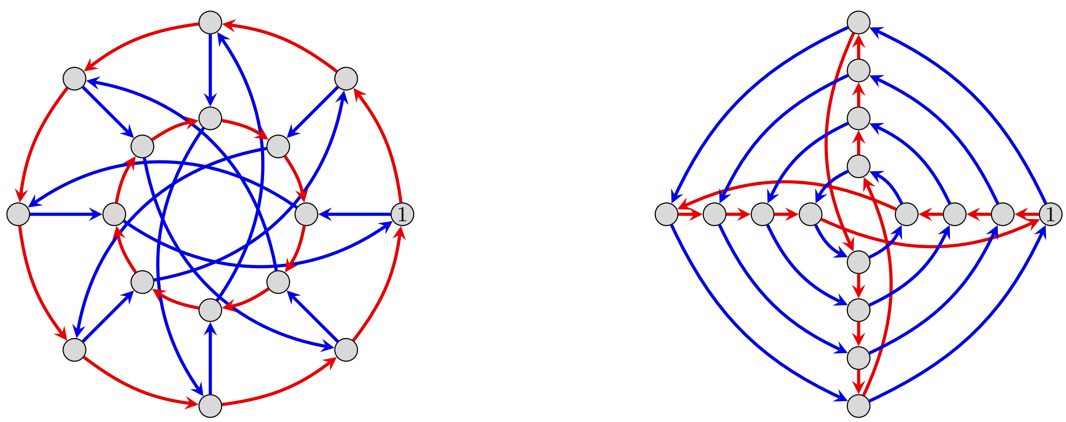
1. For each $n$, sketch the $n^{\text{th}}$ roots of unity on the unit circle, and list the primitive $d^{\text{th}}$ roots for each $d \mid n$. Then factor $x^n - 1$ as a product of irreducible polynomials.

   (a) $n = 8$ (b) $n = 9$ (c) $n = 10$ (d) $n = 16$.

2. For each $n$ from the previous problem, the set $U_n := \{k \mid 0 < k < n,\ \gcd(n,k) = 1\}$ forms a group under multiplication, where the result is taken modulo $n$. Construct a Cayley table, Cayley diagram, and determine to which familiar group it is isomorphic.

3. Below are Cayley diagrams of the *generalized quaternion group* $Q_{16} = \langle \zeta_8, j \rangle$, defined by replacing $\zeta_4 = e^{2\pi i/4} = i$ with $\zeta_8 = e^{2\pi i/8} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$ in the quaternion group $Q_8$.




   (a) Draw these diagrams and label each node as $a + bi + cj + dk$. Then re-draw them with each node labeled as either $\pm\zeta^m$ or $\pm\zeta^m j$, where $\zeta = \zeta_8$ and $m = 0, 1, 2, 3$.

   (b) Identifying elements of $Q_{16}$ with their negatives defines a group on 8 elements:

$$\pm 1,\ \ \pm\zeta,\ \ \pm\zeta^2,\ \ \pm\zeta^3,\ \ \pm j,\ \ \pm\zeta j,\ \ \pm\zeta^2 j,\ \ \pm\zeta^3 j.$$

   Construct a Cayley table and Cayley diagram. Which familiar group is this?

4. For each part below, the two matrices given generate a group $G = \langle A, B \rangle$, where the binary operation is matrix multiplication. Draw a Cayley diagram for each group, write a presentation, and determine to which familiar group is it isomorphic.

   (a) $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

   (b) $A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

   (c) $A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

   (d) $A = \begin{bmatrix} e^{2\pi i/8} & 0 \\ 0 & e^{-2\pi i/8} \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

5. For the numbers below, list all abelian groups of that order by writing each one as a product of cyclic groups of prime power order. Then, determine which group it is isomorphic to of the form $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$, where $n_{i+1} \mid n_i$.

   (a) $32 = 2^5$ (b) $36 = 2^2 \cdot 3^2$ (c) $400 = 2^4 \cdot 5^2$ (d) $p^3 q$; primes $p \neq q$