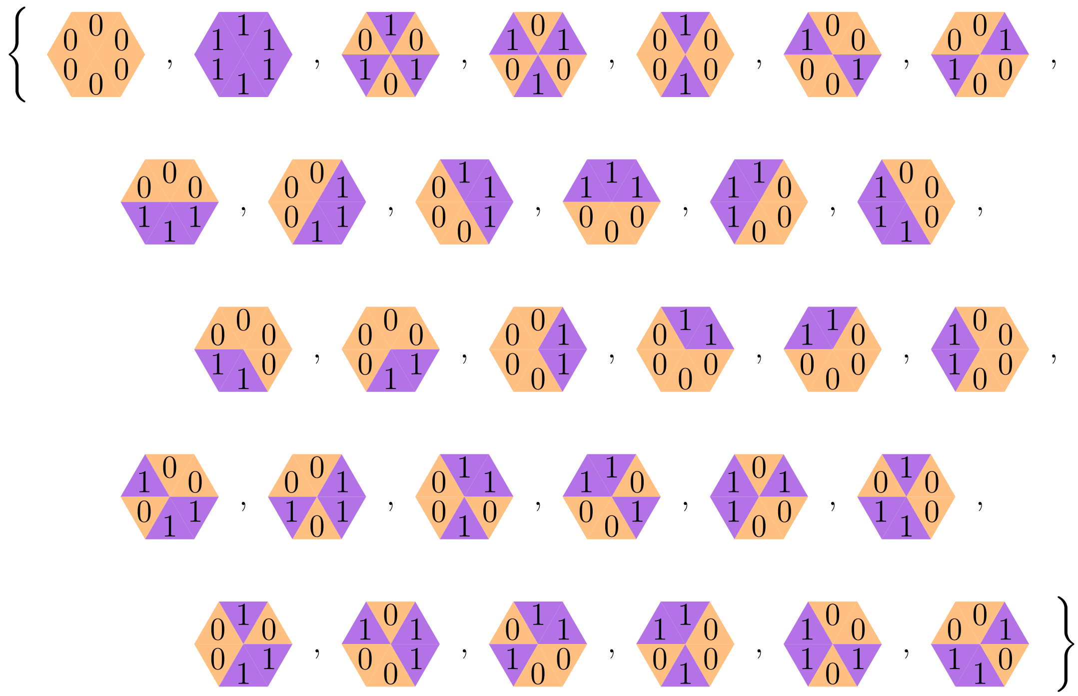
1. Consider the right action of $G = D_6 = \langle r, f \rangle$ on following set of 31 "binary hexagons," where r rotates each one 60° counterclockwise, and f flips each one horizontally (i.e., across a vertical axis).

$S =$ 

   (a) Draw the action graph.

   (b) Construct the "*fixed point table*", which has a checkmark in row g and column s if $\phi(g).s = s$.

   (c) Next to each $s \in S$ on your action graph, write $\operatorname{stab}(s)$, the stabilizer subgroup, using its generators. Which subgroups of D_6 don't appear?

   (d) The *fixator* of each $g \in D_6$, denoted $\operatorname{fix}(g)$, can be read off of the the fixed point table. What is the average size $|\operatorname{fix}(g)|$?

   (e) Find $\operatorname{Ker}(\phi)$ and $\operatorname{Fix}(\phi)$.

2. Suppose that G acts on S via the homomorphism $\phi \colon G \to \operatorname{Perm}(S)$.

   (a) Show that $\operatorname{stab}(s)$ is a subgroup for all $s \in S$. Use the notational conventions that we have been using in lecture.

   (b) Show that the stablizers of any two elements in the same orbit are conjugate – specifically, that $\operatorname{stab}(s.\phi(g)) = g^{-1}\operatorname{stab}(s)g$ for all $g \in G$ and $s \in S$. This relationship is summarized by the following commutative diagram.

$$\begin{tikzcd}
s \arrow[r, mapsto, "\phi(x)"] \arrow[d, mapsto, "\phi(g)"'] & s \arrow[d, mapsto, "\phi(g)"] \\
s' \arrow[r, mapsto, dashed, "\phi(g^{-1}xg)"] & s'
\end{tikzcd}$$

3. Suppose a group G of order 55 acts on a set S of size 14. Let $s \in S$ be an arbitrary element.

   (a) What are the possible sizes of the orbit of s?

   (b) What are the possible sizes of the stabilizer of s?

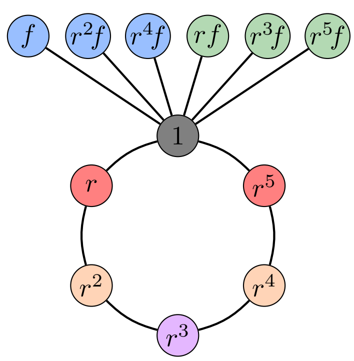
   (c) Show that this action must have a fixed point.

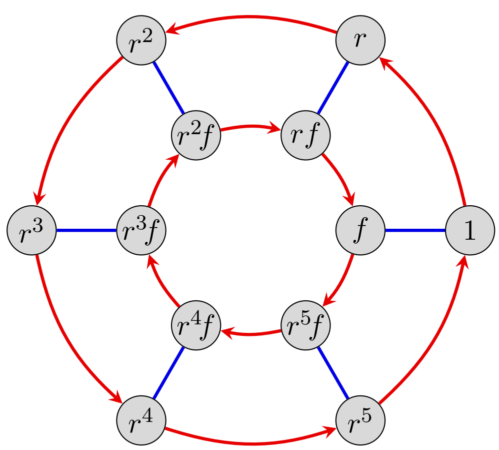
   (d) What is the fewest number of fixed points that this action can have? Justify your answer.

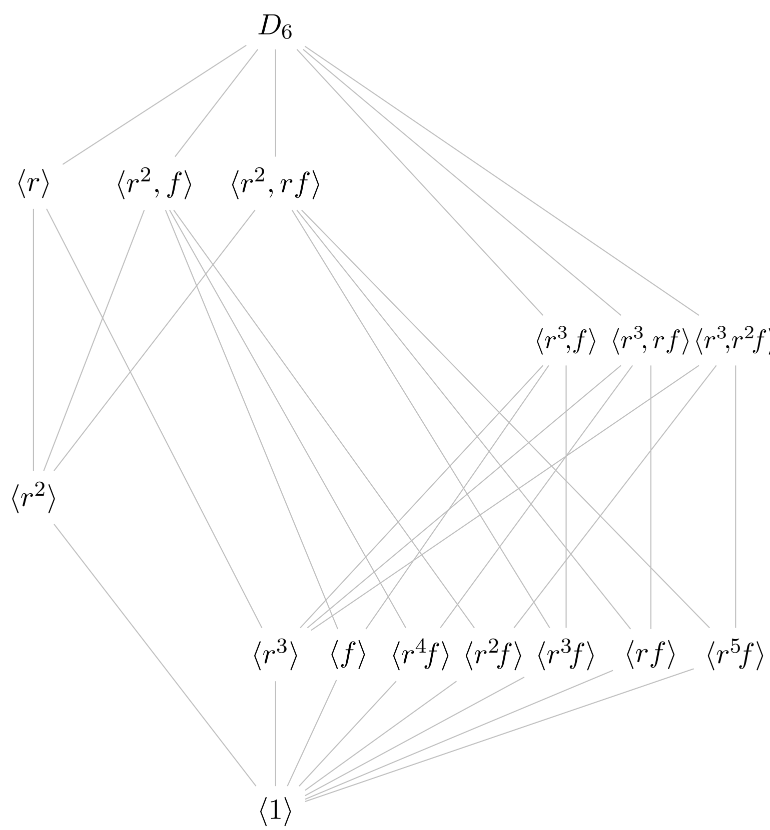
4. Let $G = D_6 = \langle r, f \rangle$ act on its set $S = \{H \leq D_6\}$ of subgroups by conjugation, i.e.,

$$\phi\colon G \longrightarrow \mathrm{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } H \mapsto g^{-1}Hg.$$

   A Cayley graph, cycle graph, and subgroup lattice for D_6 are shown below.







   (a) Construct the action graph, and superimpose it on the subgroup lattice.

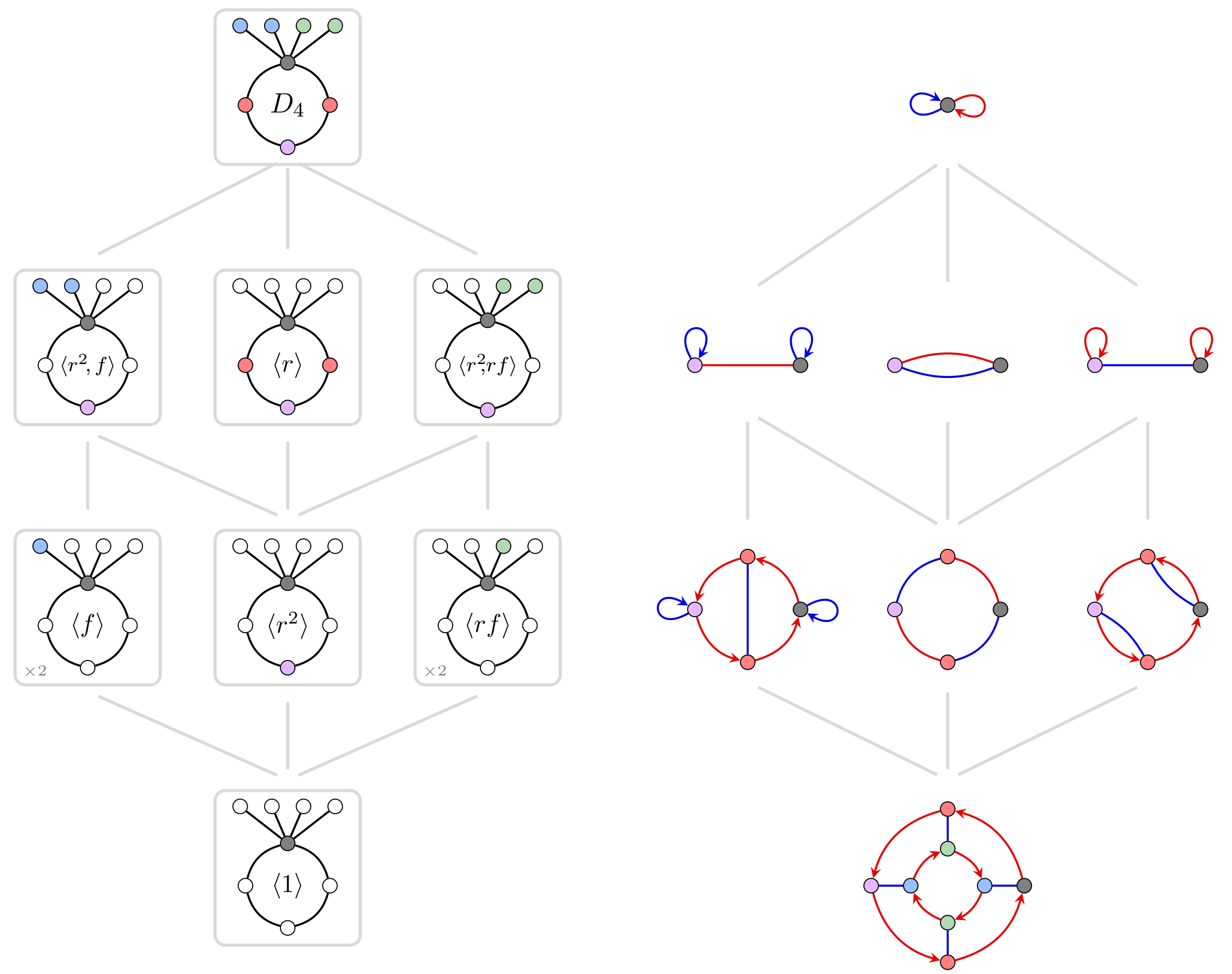
   (b) Construct the fixed point table.

   (c) Find $\mathrm{stab}(H)$ for each subgroup $H \leq D_6$, and $\mathrm{fix}(g)$ for each $g \in D_6$.

   (d) Find $\mathrm{Ker}(\phi)$ and $\mathrm{Fix}(\phi)$.

   (e) Interpret $\mathrm{orb}(H)$, $\mathrm{stab}(H)$, $[G : \mathrm{stab}(H)]$, $\mathrm{Fix}(\phi)$, $\mathrm{Ker}(\phi)$, $\mathrm{fix}(g)$, and the average size of $|\,\mathrm{fix}(g)|$ in terms of familiar algebraic objects.

5. There is a *Galois correspondence* between conjugacy classes of subgroups of G and transitive actions of G, defined by collapsing the right cosets of H. An example of this correspondence for D_4 is shown below.



(a) Carry out this construction for the group $D_6 = \langle r, f\rangle$.

(b) Repeat the previous exercise for $D_6 = \langle s, t\rangle$, where $s = f$ and $t = fr$. Label each conjugacy class with a subgroup that contains it, using this generating set.