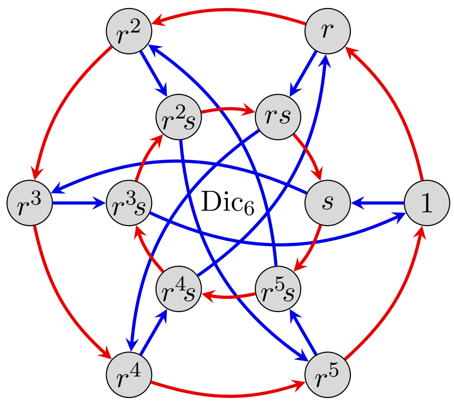
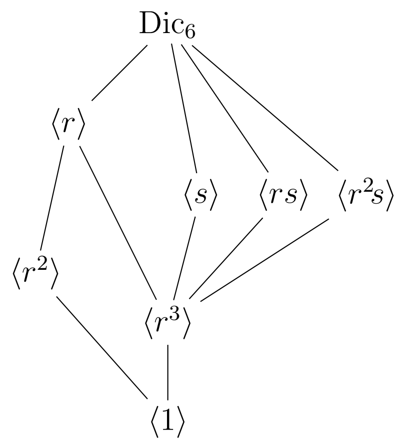
1. In this problem, we will explore the actions of the dicyclic group Dic_6 and its automorphism group on itself and its subgroups by conjugation. A Cayley diagram, subgroup lattice, and conjugacy classes are shown below.





rs	r^5s	r^3s
s	r^2s	r^4s
r^3	r^2	r^4
1	r	r^5

(a) The right action of Dic_6 on itself by conjugation is defined by the homomorphism

$$\phi\colon \text{Dic}_6 \longrightarrow \text{Perm}(S), \qquad \phi(g) = \text{the permutation that sends each } x \mapsto g^{-1}xg.$$

Draw the action diagram and construct the fixed point table. Find $\text{stab}(s)$ for each $s \in S$, $\text{fix}(g)$ for each $g \in G$, as well as $\text{Ker}(\phi)$ and $\text{Fix}(\phi)$.

(b) The inner automorphism group

$$\text{Inn}(\text{Dic}_6) \cong \text{Dic}_6 / Z(\text{Dic}_6) = \text{Dic}_6 / \langle r^3 \rangle \cong D_3$$

also acts on Dic_6. Construct the action diagram and fixed point table of this action and find $\text{stab}(s)$, $\text{fix}(g)$, $\text{Ker}(\phi)$ and $\text{Fix}(\phi)$. Then draw the subgroup lattice of $\text{Inn}(\text{Dic}_6) = \langle \varphi_r, \varphi_s \rangle$, where $\varphi_g \colon x \mapsto g^{-1}xg$.

(c) The automorphism group of Dic_6 is $\text{Aut}(\text{Dic}_6) = \langle \varphi_r, \varphi_s, \omega \rangle$ acts on Dic_6, where ω is the outer automorphism defined by

$$\omega\colon \text{Dic}_6 \longrightarrow \text{Dic}_6, \qquad \omega(r) = r, \quad \omega(s) = s^{-1} = r^3s,$$

that "reverses" the blue arrows. Make a diagram showing how each automorphism permutes the elements of Dic_6. Then construct the action diagram, fixed point table, and find $\text{stab}(s)$, $\text{fix}(g)$, $\text{Ker}(\phi)$ and $\text{Fix}(\phi)$.

(d) The automorphism group $\text{Aut}(\text{Dic}_6) = \langle \varphi_r, \varphi_s, \omega \rangle$ is isomorphic to D_6. Construct a Cayley diagram and subgroup lattice using these generators.

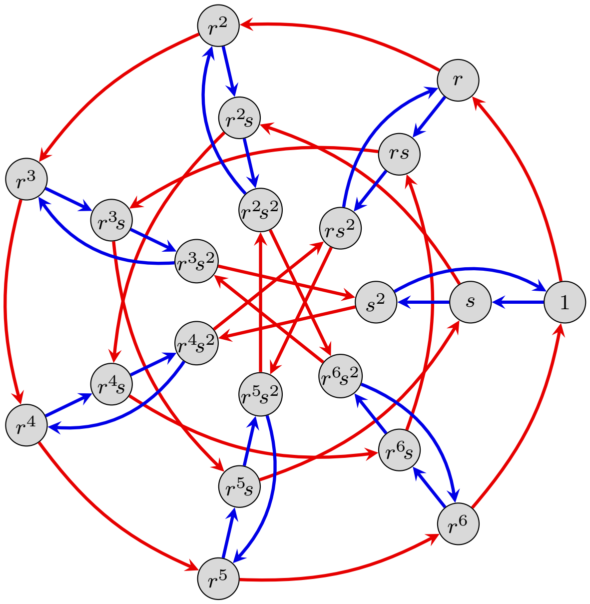
(e) The group $\text{Aut}(\text{Dic}_6)$ also acts on the conjugacy classes of Dic_6. Construct the action diagram, fixed point table, and find $\text{stab}(s)$, $\text{fix}(g)$, $\text{Ker}(\phi)$ and $\text{Fix}(\phi)$.

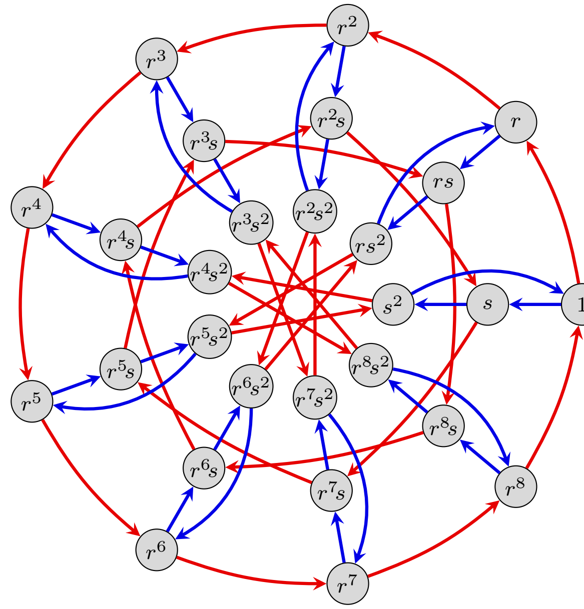
(f) The group Dic_6 acts on its subgroups by conjugation, via the homomorphism

$$\phi\colon \text{Dic}_6 \longrightarrow \text{Perm}(S), \qquad \phi(g) = \text{the permutation that sends each } H \mapsto g^{-1}Hg.$$

Construct the action diagram superimposed on the subgroup lattice. Then construct the fixed point table and find $\text{stab}(s)$, $\text{fix}(g)$, $\text{Ker}(\phi)$ and $\text{Fix}(\phi)$.

2. Carry out the following steps for the groups $C_7 \rtimes C_3$ and $C_9 \rtimes C_3$, whose Cayley diagrams are shown below.





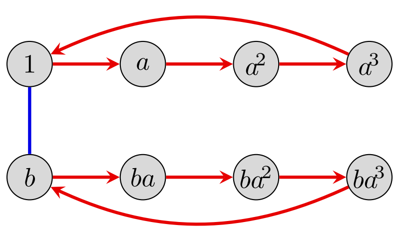
(a) Let G act on its subgroups by conjugation. Draw the action diagram superimposed on the subgroup lattice. Construct the fixed point table, and find $\text{stab}(H)$ for each $H \leq G$, $\text{Ker}(\phi)$ and $\text{Fix}(\phi)$.

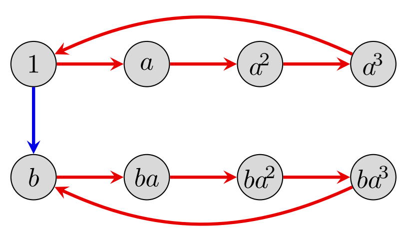
(b) Let G act on the right cosets of $H = \langle s \rangle$, via the homomorphism

$$\phi\colon G \longrightarrow \text{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } Hx \mapsto Hxg.$$

Construct the action diagram, fixed point table, and find $\text{stab}(Hx)$ for each right coset, $\text{Ker}(\phi)$ and $\text{Fix}(\phi)$.

3. Let G be an unknown group of order 8. If it has no element of order 4, then $g^2 = e$ for all $g \in G$, and so G must be abelian. Otherwise, it has a "partial Cayley diagram" like one of the following:





Find all possibilities for finishing each diagram, and label by isomorphism type.

4. Let $\phi\colon G \to \text{Perm}(S)$ be a *left* group action. Prove the orbit-stabilizer theorem by constructing a bijection between $\text{orb}(s)$ and *left* cosets of $H = \text{stab}(s)$. Use analogous notational conventions from lecture, e.g., $\phi(g).s$ instead of $s.\phi(g)$.