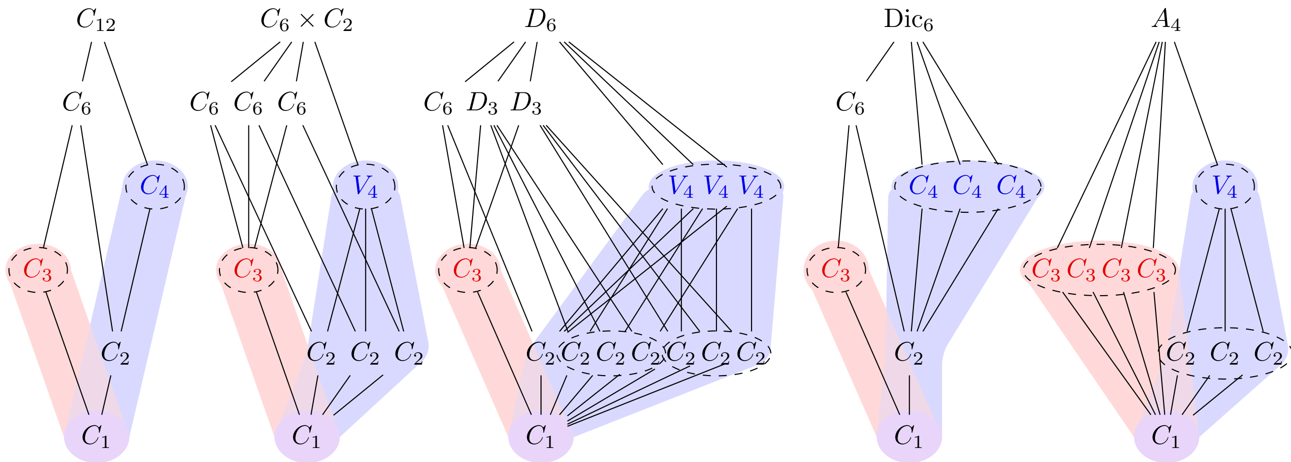
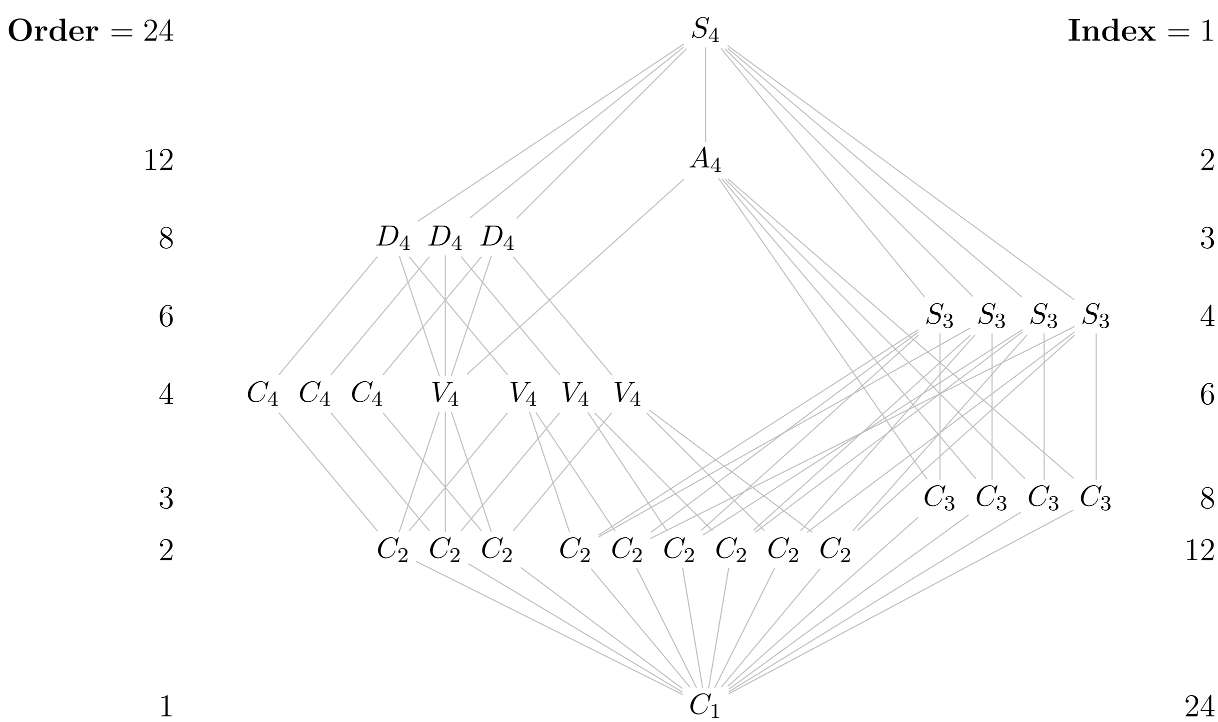
1. Loosely speaking, the Sylow theorems tell us that (1) all p-subgroups come in a single "p-subgroup tower", (2) the "top" of these towers are a single conjugacy class, and (3) the size of this class is 1 mod p. This is illustrated below with the groups of order 12.



Using the LMFDB, construct analogous diagrams for the groups of order 18 and 20.

2. The subgroup lattice of the symmetric group S_4 is shown below.



   (a) Partition the subgroups into conjugacy classes. Carefully and completely justify your answers using the Sylow theorems, without making reference to cycle type.

   (b) For each conjugacy class $\mathrm{cl}_G(H)$, find the isomorphism type of the normalizer $N_G(H)$.

   (c) Using the GroupNames website, make a table of all 15 groups of order 24, the number of subgroups, and basic information about their Sylow p-subgroups (number and isomorphism type). Write down at least one observation that you find interesting.

(d) Which groups are *not* an internal direct or semidirect product of Sylow subgroups?

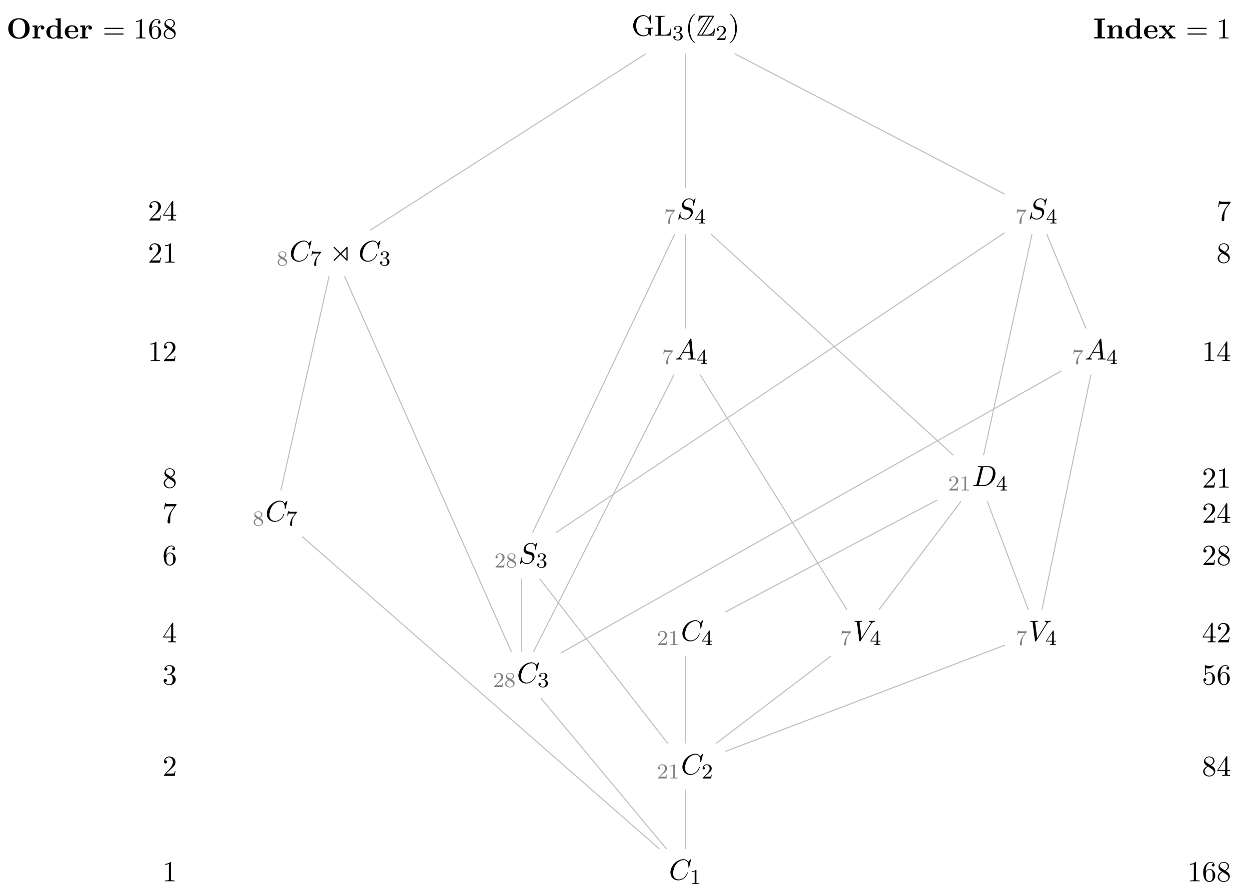
(e) None of the following groups are among the 15 listed on GroupNames: $D_6 \times C_2$, $C_6 \rtimes C_4$, $C_6 \rtimes C_2^2$, $C_4 \rtimes C_6$, $C_3 \rtimes C_2^3$, $C_2^3 \rtimes C_3$, $C_2^2 \rtimes C_6$, $C_3 \rtimes Q_8$, $Q_8 \rtimes C_3$, $C_4 \rtimes S_3$. Find which of the 15 each is isomorphic to, and add this this to your table under the "alias(es)" column.

3. Show that there are no simple groups of the following order.

(i) $45 = 3^2 \cdot 5$ (ii) $56 = 2^3 \cdot 7$ (iii) $108 = 2^2 \cdot 3^3$ (iv) $p^n \quad (n > 1)$.

[*Hint*: For Part (d), first use a suitable group action to show that $|Z(G)| > 1$.]

4. After A_5, the next smallest nonabelian simple group is $G = \mathrm{GL}_3(\mathbb{Z}_2)$, the invertible 3×3 binary matrices. It has order $168 = 2^3 \cdot 3 \cdot 7$, and its conjugacy poset is shown below.



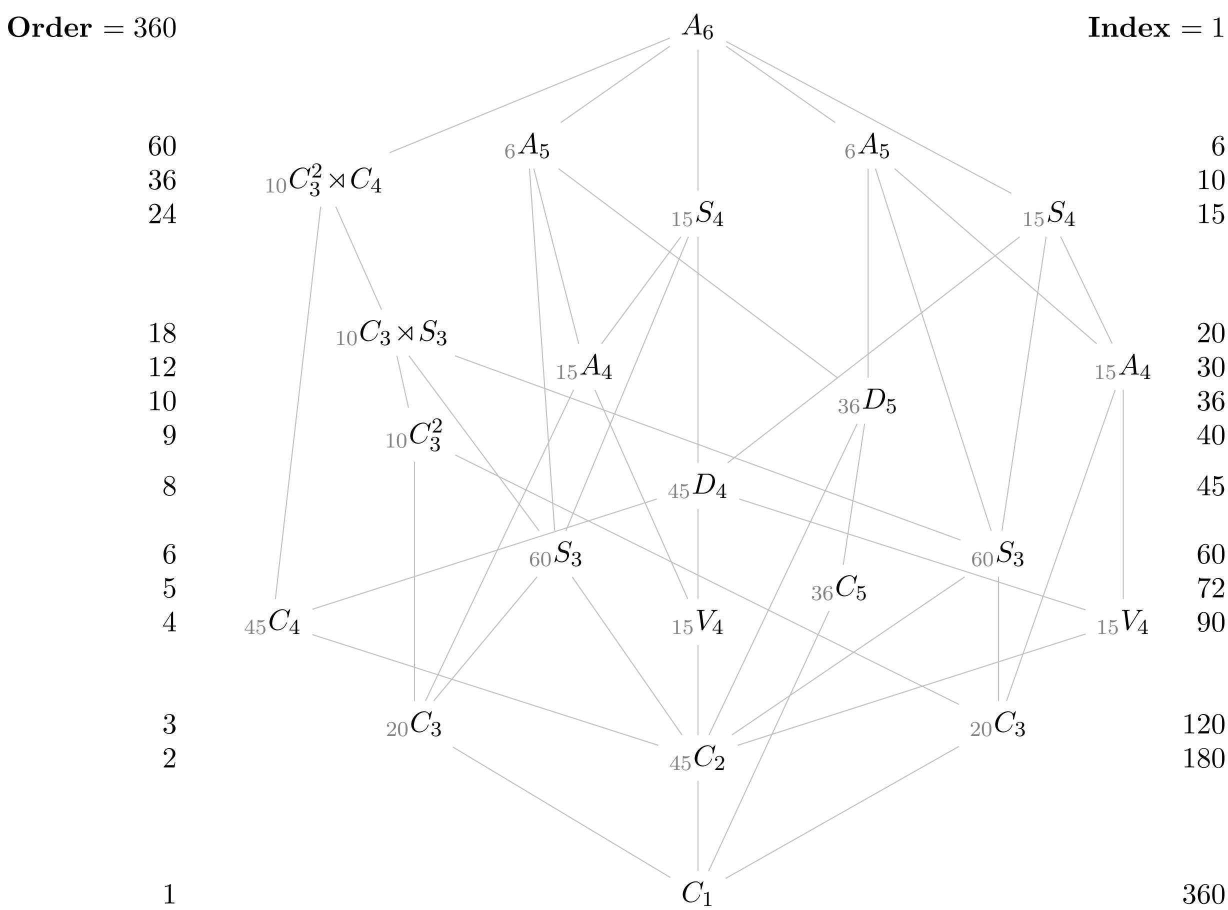
(a) Color-code the p-subgroups, then draw arrows from each $\mathrm{cl}(H)$ to $\mathrm{cl}(N(H))$.

(b) Show that there is a non-trivial homomorphism $\phi\colon \mathrm{GL}_3(\mathbb{Z}_2) \to S_8$.

(c) Show that this homomorphism must be an embedding, and conclude that the order-40320 group S_8 has at least one subgroup isomorphic to $\mathrm{GL}_3(\mathbb{Z}_2)$.

(d) Show that every such subgroup of S_8 additionally must be contained in A_8.

5. The alternating group A_6 is the third smallest nonabelian simple group. It has order $6!/2 = 360 = 2^3 \cdot 3^2 \cdot 5$, and 501 subgroups contained in 22 conjugacy classes.



(a) Distinguish the p-subgroups by colors on the lattice.

(b) For each non-singleton conjugacy class $\mathrm{cl}(H)$, draw an arrow from it to $\mathrm{cl}(N(H))$, the conjugacy class of its normalizer.

(c) Now, let G be an unknown group of order $90 = 2 \cdot 3^2 \cdot 5$.

(i) Show that if G has a non-normal Sylow 5-subgroup, then there is be a non-trivial homomorphism $\phi\colon G \to S_6$.

(ii) Show that if $\phi(G)$ is contained in the simple group A_6, then ϕ cannot be injective.

(iii) Explain why this implies that G cannot be simple.

(iv) Find all possibilities for n_2, n_3, and n_5, where n_p is the number of Sylow p-subgroups of G. Then, using GroupNames or LMFDB, make a list of all groups of order 90, and write down the actual vaules of n_2, n_3, and n_5 for each, as well as the isomorphism type of the Sylow 3-subgroup(s) – either C_9 or C_3^2. Does anything surprise you about this?