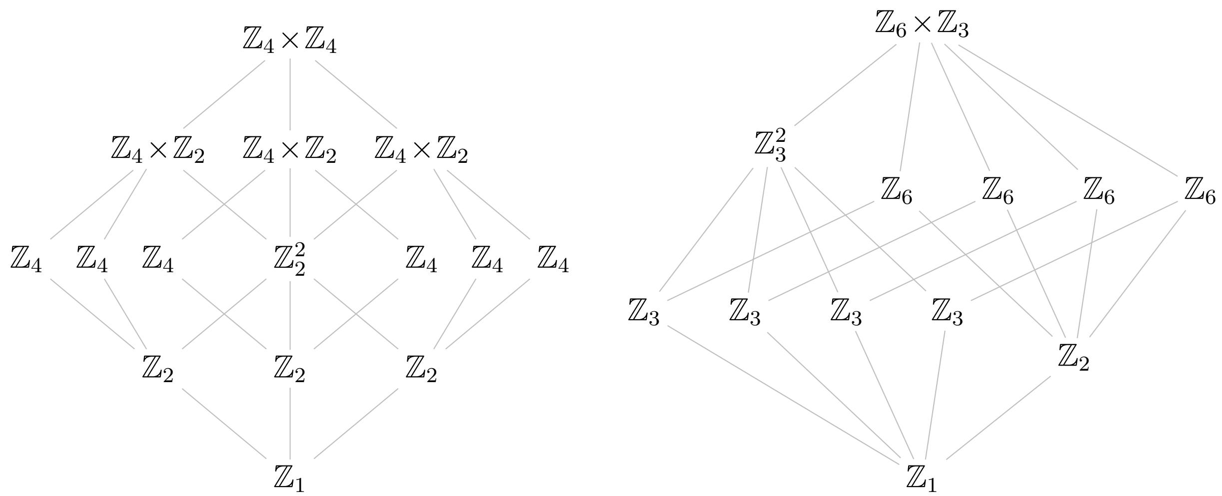
1. The subgroup lattices of $\mathbb{Z}_4 \times \mathbb{Z}_4$ and $\mathbb{Z}_6 \times \mathbb{Z}_3$ are shown below. Re-draw these lattices with the subgroups written with generator(s), and then construct their *subring lattices* by coloring each subgroup based on whether it is an ideal, subring but not an ideal, or subgroup that is not a subring.



Finally, for each ring, write down the units, and the zero divisors.

2. All of the isomorphism theorems for groups have analogues for rings. The proofs just amount to showing that the group homomorphisms are additionally ring homomorphisms, and we will carry out these details in this problem.

   (a) The *fundamental homomorphism theorem* (FHT) says that $R/\operatorname{Ker}(\phi) \cong \operatorname{Im}(\phi)$. To prove this, we construct a map
   $$\iota \colon R/I \longrightarrow \operatorname{Im}(\phi), \qquad \iota(r+I) = \phi(r),$$
   which we already know is a well-defined group isomorphism. Show that it is also a ring homomorphism.

   (b) By the *correspondence theorem*, every subgroup of R/I has the form J/I for some $I \leq J \leq R$. Show that J/I is an ideal of R/I if and only if J is an ideal of R.

   (c) The *fraction theorem* says that $(R/I)/(J/I) \cong R/J$. This is proven by constructing a map
   $$\phi \colon R/I \longrightarrow R/J, \qquad \phi(r+I) = r+J$$
   which we already know is a group homomorphism with $\operatorname{Ker}(\phi) = J/I$. Show that it is also a ring homomorphism, and then apply the FHT.

   (d) The *diamond theorem* says that $(S+I)/I \cong S/(S\cap I)$ for a subring S and ideal I.

      (i) Prove that $S \cap I$ is an ideal of S.

      (ii) We already know that
      $$\phi \colon S \longrightarrow (S+I)/I, \qquad \phi(s) = s+I$$
      is a group homomorphism with $\operatorname{Ker}(\phi) = S\cap I$. Show that it is also a ring homomorphism, and then apply the FHT.

3. Let I and J be ideals of a commutative ring R.

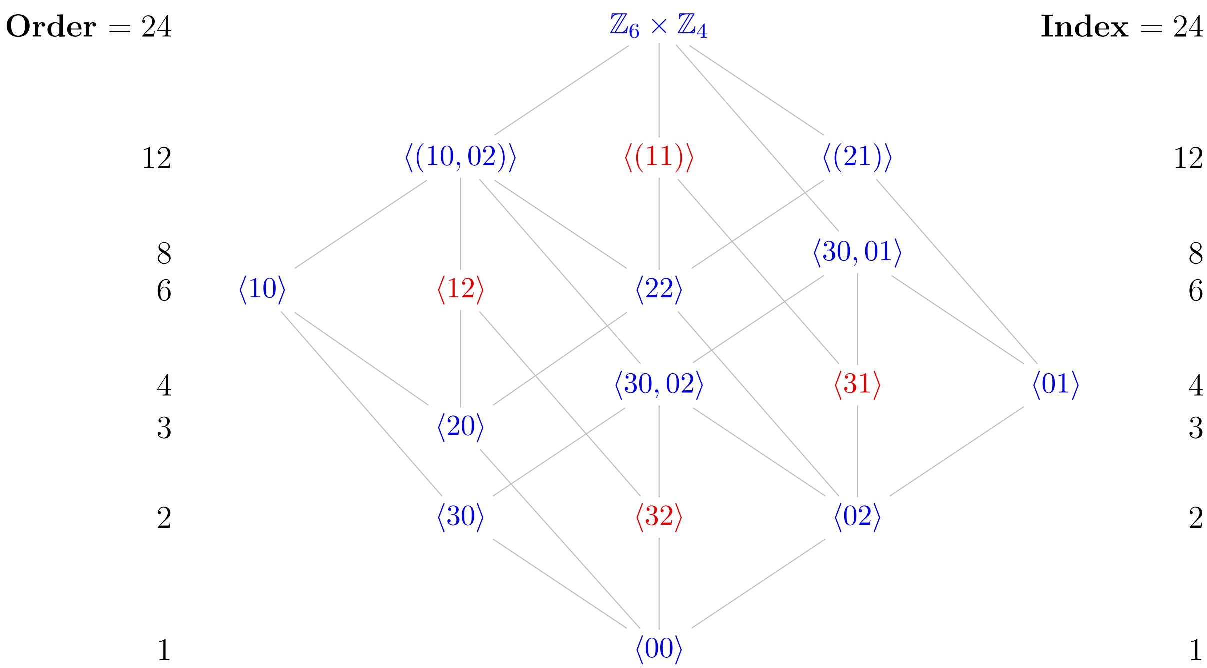
   (a) Show that $I+J$, $I\cap J$, and IJ are ideals of R. Which of these remain ideals if the commutativity hypothesis is dropped?

   (b) The set $(I:J):=\{r\in R \mid rJ\subseteq I\}$ is called the *ideal quotient* or *colon ideal* of I and J. Show that $(I:J)$ is an ideal of R. Does this require commutativity?

   (c) Determine $I+J$, $I\cap J$, IJ, and $(I:J)$ for the ideals $I=n\mathbb{Z}$ and $J=m\mathbb{Z}$ of $R=\mathbb{Z}$.

   (d) Repeat Part (c) for several pairs of ideals of $R=\mathbb{Z}_6\times\mathbb{Z}_4$, whose subring lattice is shown below.

   (e) Describe how to find IJ and $(I:J)$ by inspection, using only the subring lattice, if possible.



4. Let $f\colon R\to S$ be a ring homomorphism between commutative rings.

   (a) If f is surjective and I is an ideal of R, show that $f(I)$ is an ideal of S.

   (b) Show that Part (a) is not true in general when f is not surjective.

   (c) Show that if f is surjective and R is a field, then S is a field as well.

5. Let R be a commutative ring with 1.

   (a) Show that if x is contained in every maximal ideal, then $1+x$ is a unit.

   (b) A ring is *local* if it has a unique maximal ideal. Show that R is local if and only if the non-units form an ideal.

   (c) Let p be a fixed prime. Show that the ring

$$R=\left\{\tfrac{a}{b}\mid a,b\in\mathbb{Z},\ (a,b)=1,\ p\nmid b\right\}\subseteq\mathbb{Q}$$

   is local, and characterize units and maximal unique ideal.