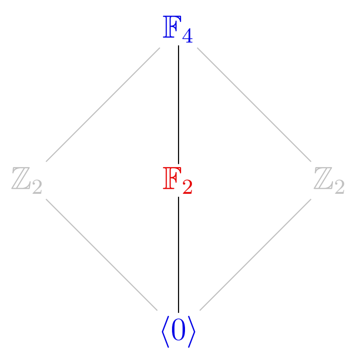
1. Construct the Cayley tables, Cayley graphs, and subring lattices of the finite field $\mathbb{F}_9 \cong \mathbb{Z}_3[x]/(x^2+x+2)$. Examples for the finite fields

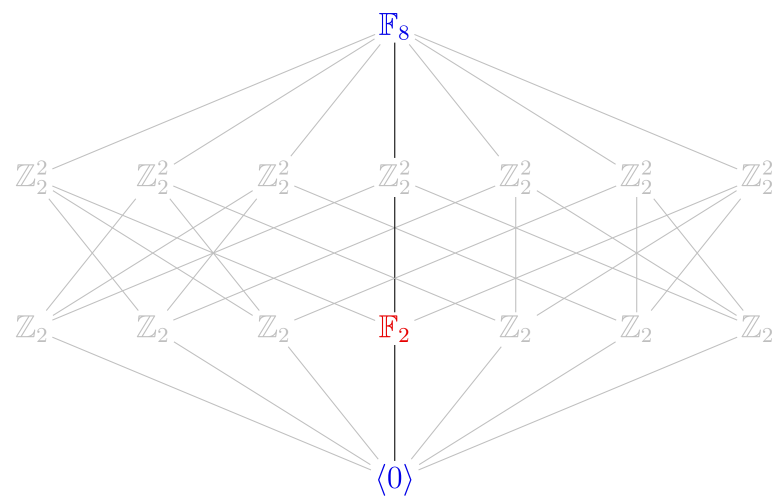
$$\mathbb{F}_4 \cong \mathbb{Z}_2[x]/(x^2+x+1) \qquad \text{and} \qquad \mathbb{F}_8 \cong \mathbb{Z}_2[x]/(x^3+x+1)$$

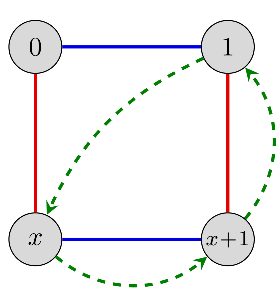
are shown below.

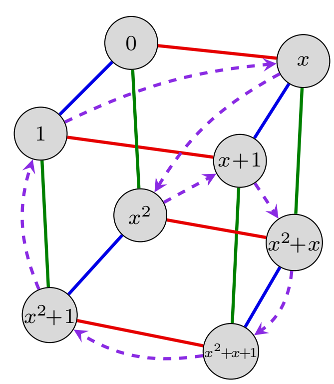
$\times$	1	x	$x+1$
1	1	x	$x+1$
x	x	$x+1$	1
$x+1$	$x+1$	1	x

$\times$	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
1	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
x	x	x^2	x^2+x	$x+1$	1	x^2+x+1	x^2+1
$x+1$	$x+1$	x^2+x	x^2+1	x^2+x+1	x^2	1	x
x^2	x^2	$x+1$	x^2+x+1	x^2+x	x	x^2+1	1
x^2+1	x^2+1	1	x^2	x	x^2+x+1	$x+1$	x^2+x
x^2+x	x^2+x	x^2+x+1	1	x^2+1	$x+1$	x	x^2
x^2+x+1	x^2+x+1	x^2+1	x	1	x^2+x	x^2	$x+1$









2. Use Zorn's lemma to show that the ring $\mathbb{R}$ contains a subring A containing 1 that is maximal with respect to the property that $1/2 \notin A$.

3. In this problem, we will explore several "radicals" of an ideal I in a commutative ring R.

   (a) The *radical* of $I \subseteq R$ is the set

   $$\sqrt{I} := \{x \in R \mid x^n \in I \text{ for some } n \in \mathbb{N}\},$$

   and I is a *radical ideal* if $\sqrt{I} = I$.

   (i) Show that $\sqrt{I}$ is an ideal containing I.

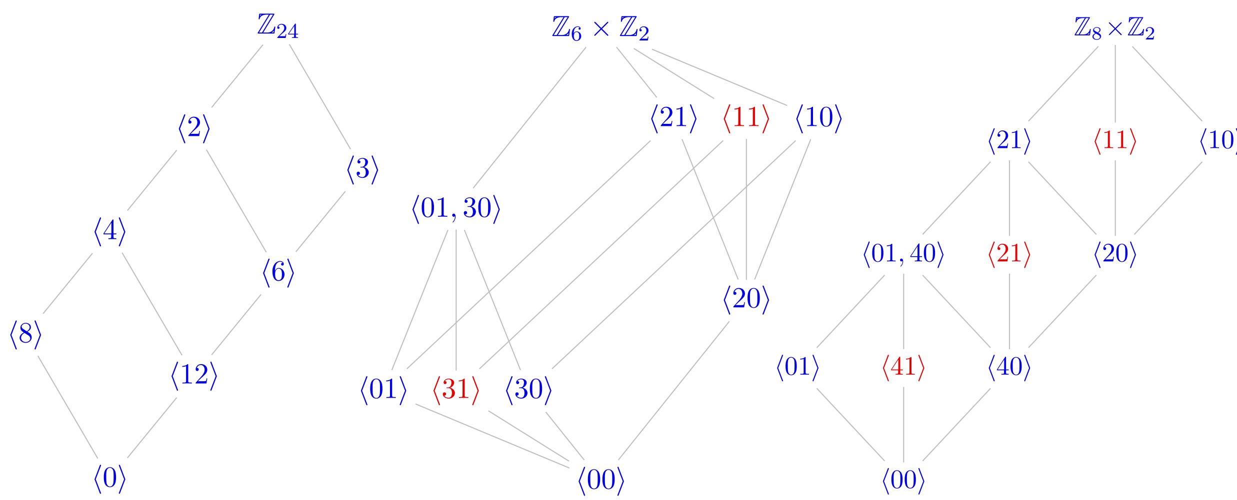
   (ii) Find the radicals of all ideals of the rings $\mathbb{Z}_6 \times \mathbb{Z}_4$, $\mathbb{Z}_8 \times \mathbb{Z}_2$, $\mathbb{Z}_6 \times \mathbb{Z}_2$, and $\mathbb{Z}_{24}$. Denote these on the subring lattices by drawing an arrow from each I to $\sqrt{I}$, and find the nilradical $\mathfrak{N}_{R/I} = \sqrt{I}/I$.

   (b) The *Jacobsen radical* of I, denoted $\text{jac}(I)$, is the intersection of all maximal ideals that contain I.

   (i) Show that $\text{jac}(I)$ is an ideal.

   (ii) Find the Jacobsen radical of all proper ideals of the rings $\mathbb{Z}_{24}$, $\mathbb{Z}_6 \times \mathbb{Z}_4$, $\mathbb{Z}_6 \times \mathbb{Z}_2$, and $\mathbb{Z}_8 \times \mathbb{Z}_2$. Denote these by drawing an arrow from I to $\text{jac}(I)$ on a fresh copy of the lattices.

   (c) The nilradical and Jacobson radical of R, denoted $\mathfrak{N}_R := \sqrt{0}$ and $\text{Jac}(R) := \text{jac}(0)$, are the intersection of the prime and maximal ideals, respectively. Mark these on the subring lattices.



4. Let R be a commutative ring with 1. An ideal $I \subsetneq R$ is *primary* if $ab \in I$ implies $a \in I$ or $b^n \in I$ for some $n \in \mathbb{N}$, and *prime* if this holds for $n = 1$.

   (a) Show that an ideal I is prime if and only if it is primary and radical.

   (b) Show that the radical of a primary ideal is prime.

   (c) Suppose that R is a PID. Characterize its nonzero primary ideals.

   (d) Show that I is primary if and only if all zero-divisors in R/I are nilpotent.

   (e) Find an example of a primary ideal that is not generated by a prime power power.