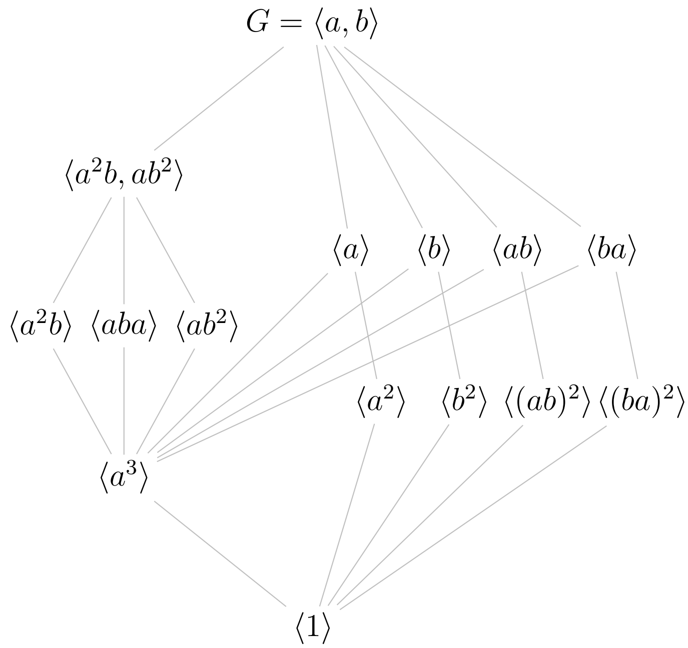
Math 8510, Final Exam. December 12, 2022

1. (8 points) Give a formal definition of what it means for a set G to be a *group*.

2. (10 points) Draw the subgroup lattice of the dihedral group D_7. Write the subgroups by generator(s), label the edges by the corresponding index, and denote the conjugacy classes by dashed circles.

3. (8 points) Let G be a group of order 30. Show that if G is not cyclic, then it is nonabelian. State any results about the structure of groups that you use.

4. (36 points) In this problem, G refers to the special linear group $\mathrm{SL}_2(\mathbb{Z}_3)$ of order 24, whose subgroup lattice appears below. *For full credit, you must briefly justify your answers to Parts (c)–(i).*



(a) Used dashed lines to partition the fifteen subgroups into conjugacy classes.

(b) Find the normalizer of the following subgroups:

$N_G(\langle a \rangle) = \qquad$, $N_G(\langle a^2 \rangle) = \qquad$, $N_G(\langle a^3 \rangle) = \qquad$, $N_G(\langle a^2b \rangle) = \qquad$, $N_G(\langle a^2b, ab^2 \rangle) = \qquad$.

(c) What familiar group is the quotient $G/\langle a^3 \rangle$ isomorphic to, and why?

(d) What familiar group is the subgroup $\langle a^2b, ab^2 \rangle$ isomorphic to, and why?

(e) What familiar group is the quotient $\langle a^2b, ab^2\rangle/\langle a^3\rangle$ isomorphic to, and why?

(f) Decide whether or not G can be written as a nontrivial direct or semidirect product.

(g) Find the center, $Z(G)$, and justify your answer. [*Hint*: Recall that this is a group of matrices.]

(h) Is G solvable? Nilpotent? Why or why not?

(i) Determine what $\mathrm{Inn}(G)$ (the inner automorphism group) is isomorphic to, with justification.

5. (36 points) Let G be a finite group.

   (a) Show that if $H \leq G$ and $[G : H] = 2$, then $H \trianglelefteq G$. Does this also hold for infinite groups? Why or why not?

   (b) Let G act on its subgroups by conjugation, via

   $$\phi\colon G \to \mathrm{Perm}(S), \qquad \phi(g) = \text{the permutation sending each } H \text{ to } g^{-1}Hg.$$

   Describe $\mathrm{orb}(H)$, $\mathrm{stab}(H)$, $\mathrm{Ker}(\phi)$, and $\mathrm{Fix}(\phi)$, in plain English, using familiar group-theoretic terms when possible.

   (c) Carefully state the orbit-stabilizer theorem. Your answer should read as if it were in a textbook.

(d) Show that the size of any conjugacy class $\mathrm{cl}_G(H)$ divides $|G|$.

(e) Show that if G has a subgroup with exactly two conjugates (itself included), then it is not simple. You may use the results of previous part(s) even if you could not prove them.

(f) Does the previous statement hold if "subgroup" is replaced with "element"? Why or why not?

6. (12 points) Use the Sylow theorems to show that there are no simple groups of order $56 = 2^3 \cdot 7$.

7. (12 points) For each of the following, determine if it is true or false. If true, prove it. If false, provide a counterexample.

   (a) $H \trianglelefteq N_G(H)$

   (b) $N_G(H) \trianglelefteq G$.

8. (12 points) Let R be a commutative ring with 1. Show that an ideal P is prime iff R/P is an integral domain.

9. (10 points) Let R be a commutative ring with 1. Show that every maximal ideal is prime. You may known results about maximal ideals provided you properly state them.

10. (30 points) Recall that a group G is *solvable* if $G^{(k)} := [G^{(k-1)}, G^{(k-1)}] = 1$ for some k. The minimal k is called its *solvable length*. Let **Svl** be the category of solvable groups, and let $\mathbf{Svl}_{\leq n}$ be the category of solvable groups of length at most n.

    (a) Prove that if $f\colon G \to H$ is a surjection between solvable groups, then the solvable length of G is at least the solvable length of H.

    (b) Carefully state what it would mean for free objects to always exist in the category $\mathbf{Svl}_{\leq n}$. Include a commutative diagram that illustrates this.

    (c) Carefully state what it would mean for free objects to always exist in the category **Svl**. Include a commutative diagram that illustrates this.

(d) Prove or disprove that free objects always exist in **Svl**.

(e) Prove or disprove that free objects always exist in $\mathbf{Svl}_{\leq n}$.

11. (20 points) Give an example of each of the following (no justification needed), or say "does not exists" (DNE), if there is none.

    (a) An nonabelian group such that every proper subgroup is normal.

    (b) A group G with isomorphic subgroups $H_1 \cong H_2$ such that $G/H_1 \not\cong G/H_2$.

    (c) An outer automorphism of a group. [Define it by where the generators get sent.]

    (d) A nonabelian group for which $\mathrm{Inn}(G) = \mathrm{Aut}(G)$.

    (e) Two nonisomorphic non-cyclic abelian groups of order 24.

    (f) Two non-isomorphic subgroups of D_4 of the same order. [Give generating sets.]

    (g) A chain of subgroups $K \trianglelefteq H \trianglelefteq G$ for which $K \ntrianglelefteq G$. [Give generating sets.]

    (h) A proper subgroup H of a group G such that $N_G(H) = H$.

    (i) A subgroup H of a group G such that $H \lneq N_G(H) \lneq G$.

    (j) A nontrivial group G whose commutator subgroup G' is itself.

    (k) Two elements in S_5 of the same order that are not conjugate.

    (l) A non-simple, non-solvable group.

    (m) A group that is nilpotent but not solvable.

    (n) A non-principal ideal.

    (o) An integral domain that is not a UFD.

    (p) A UFD that is not a PID.

    (q) A Euclidean domain that is not a field.

    (r) A maximal ideal of the polynomial ring $\mathbb{Z}[x]$.

    (s) A maximal ideal I of the ring $R = \mathbb{F}_3[x]$ for which $R/I \cong \mathbb{F}_{27}$.

    (t) An irreducible element of a commutative ring that is not prime.

12. (6 points) What was your favorite topic in the class? Specifically, what did you find the most interesting, and why?