Math 8510, Midterm 1. October 12, 2022

1. (6 points) Let G be a group with a subgroup $H = \langle b, c\rangle$. No justification necessary, but *be as specific as possible with your answers.*

   (a) If $a \in H$, then what is $\langle a, b, c\rangle$?

   (b) If $a \notin H$ and $[G : H] = 3$, then what is $\langle a, b, c\rangle$?

   (c) If $a \notin H$, and $|G| = 48$ and $|H| = 6$, what are the possible orders of the subgroup $\langle a, b, c\rangle$?

2. (6 points) Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group and $V_4 = \{e, v, h, vh\}$ be the Klein 4-group. Define a homomorphism $\phi \colon Q_8 \to V_4$ by $\phi(i) = v$ and $\phi(j) = h$. Find the image of the remaining six elements.

$\phi(1) =$, $\phi(-1) =$, $\phi(k) =$, $\phi(-i) =$, $\phi(-j) =$, $\phi(-k) =$.

3. (8 points) Draw the subgroup lattice of the dihedral group $D_3 = \langle r, f \mid r^3 = f^2 = 1,\ rf = fr^2\rangle$, with the subgroups listed by generators. Label each edge between $K \leq H$ with the corresponding index, $[H : K]$. Then partition the subgroups into conjugacy classes by circling them.

4. (8 points) Prove that there are no simple groups of order $|G| = 63 = 3^2 \cdot 7$.

5. (8 points) Let G be a group with center $Z = Z(G)$. Show that Z is a subgroup of G and that it is normal.

6. (8 points) Suppose a group G of order 35 acts on a set S of size 9. Show that there must be a fixed point. Can you say something stronger?

7. (6 points) Finish the following definitions. Make sure you use terminology like "for all", where appropriate.

   (a) The *order* of an element $g \in G$ is...

   (b) A *homomorphism* ϕ from a group G to H is...

   (c) If $N \trianglelefteq G$, then the *quotient group* G/N is...

   (d) The *kernel* of a homomorphism ϕ from G is...

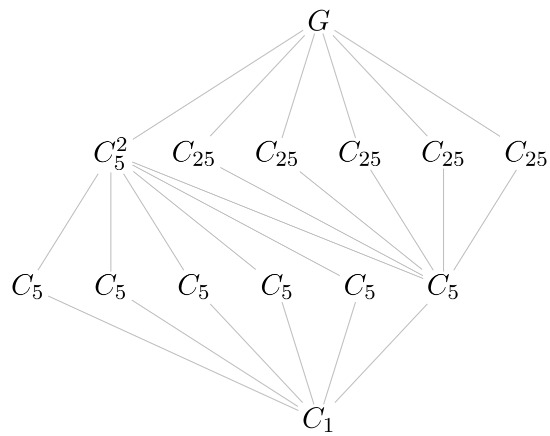
   (e) A subgroup N of G is *normal* if...

   (f) The *commutator* $[x, y]$ of elements $x, y \in G$ is...

8. (16 points) Fill in the following blanks.

   1. The smallest nonabelian group is ____________________.
   2. A list of the groups of order 8 (up to isomorphism) is ____________________.
   3. The permutation $(123)(456789) \in S_9$ has order _____, and its inverse is __________.
   4. The alternating group A_3 consists of the following elements: ____________________.
   5. Two permutations in S_n are conjugate if and only if ____________________.
   6. For any $n \geq 3$, $D_n \cong A \rtimes B$, a semidirect product of $A =$__________ with $B =$ __________.
   7. An example of a minimal generating set of S_5 of minimum size is ____________________.
   8. The group $\mathrm{Aut}(\mathbb{Z})$ has order ____________________.
   9. Let $H \leq G$ have index $[G : H] = n$. If G acts on the cosets of H by right-multiplication, then the action has ____________________ orbit(s).
   10. The kernel of a group action is the intersection of all ____________________.
   11. The group D_5 has __________ Sylow 2-subgroups, all of order __________.
   12. The kernel of the homomorphism $\phi\colon G \to \mathrm{Inn}(G)$, defined by $x \mapsto \varphi_x$ is __________.
   13. If G' is the commutator subgroup, then G/G' is the largest ____________________ of G.

9. (16 points) Answer the following about the *extraspecial group* $G = 5^{1+2}_{-}$, a nonabelian group of order $125 = 5^3$, whose subgroup lattice appears below.



(a) Let H be the "rightmost" C_5 subgroup. Explain why it must be normal.

(b) What is the quotient G/H isomorphic to, and why?

(c) Which subgroup is the normalizer, $N_G(H)$?

(d) Let K be the "leftmost" C_5 subgroup, which you may assume is not normal. What is its normalizer, $N_G(K)$?

(e) Partition the subgroups into conjugacy classes G by circling them.

(f) Is G isomorphic to a direct or semidirect product of any of its proper subgroups, and why?

(g) Mark the derived series on the lattice, i.e., write $G^{(0)} =$, $G' =$, $G'' =$,.... Is G solvable?

(h) Which subgroup must be $Z(G)$, and why? [*Hint*: G is a p-group. Also, you may use a result we mentioned in passing, but did not prove: if $G/Z(G)$ is cyclic, then G is abelian.]

(i) Determine the centralizer $C_G(x)$, where $K = \langle x \rangle$. Justify your answer.

(j) Determine the size of the conjugacy class $\mathrm{cl}_G(x)$, where $K = \langle x \rangle$.

(k) Find all N and Q (excluding G and 1) for which G is an extension of Q by N. Write each answer as an exact sequence $1 \to N \hookrightarrow G \twoheadrightarrow Q \to 1$.)

10. (8 points) Prove the diamond theorm: if $A, B \trianglelefteq G$, then $AB/A \cong B/(A \cap B)$. You may assume that $A \trianglelefteq AB$ and $(A \cap B) \trianglelefteq B$.

11. (10 points) Fix a subgroup $H \leq G$, and define the map

$$\phi\colon \big\{\text{left cosets of } N_G(H)\big\} \longrightarrow \big\{\text{conjugates of } H\big\}, \qquad \phi\colon gN_G(H) \longmapsto gHg^{-1}.$$

(a) Show that this map is well-defined and a bijection.

(b) Derive the formula $|G| = |\operatorname{cl}_G(H)| \cdot |N_G(H)|$, for any $H \leq G$.