1. Cayley graphs of two groups are shown below. Carry out the following steps for each group.

   (a) Write a presentation using the generators in the Cayley graph.

   (b) Find the left and right cosets of the following subgroups $H = \langle r \rangle$, $K = \langle s \rangle$, $L = \langle r^3 \rangle$.

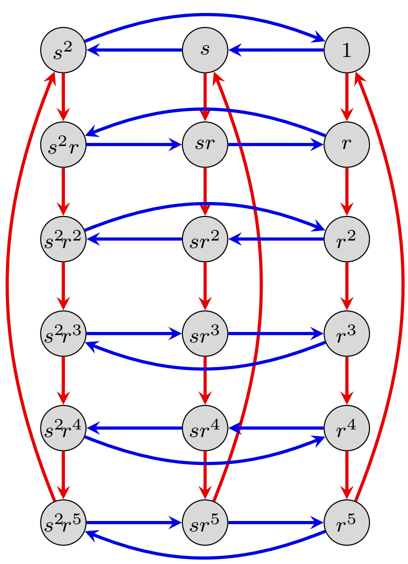
   (c) Find the normalizers of H, K, and L. Write them first as a union of cosets, and then as a subgroup by generator(s). Determine the isomorphism type of each.

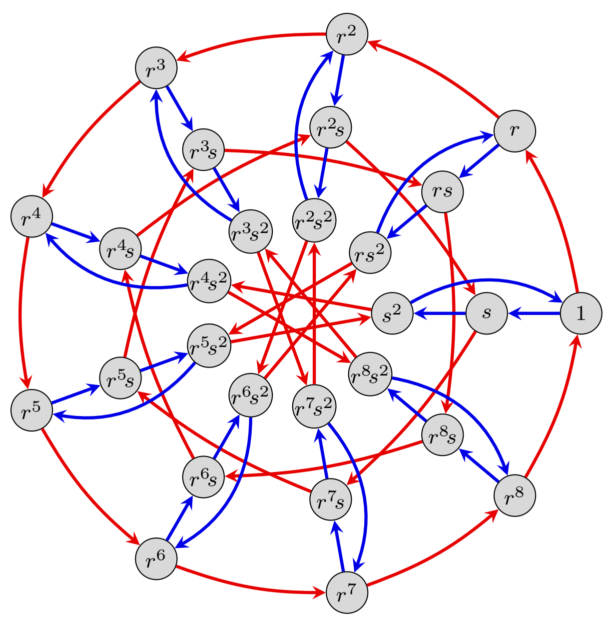
   (d) Find all conjugate subgroups of H, K, and L. Write each group by generator.

   (e) Construct the subgroup lattice, and write each subgroup with a minimal generating set. Denote the conjugacy classes of subgroups by dashed circles.

   (f) What is the center, $Z(G)$?

   (g) Find this group on the LMFDB (`https://beta.lmfdb.org/Groups/`). What is its label, and what other groups is it isomorphic to? Write down one interesting fact about it that you did not already know.





2. If $A, B \leq G$ and $x \in G$, define the (A, B)-*double coset* to be the set

$$AxB := \{axb \mid a \in A,\ b \in B\}.$$

   (a) Show that G is the disjoint union of its (A, B)-double cosets.

   (b) Show that if A and B are finite, then $|AxB| = [x^{-1}Ax : x^{-1}Ax \cap B] \cdot |B|$.

   (c) Find a Cayley graph of a group G of order at least 16, and pick two subgroups, A and B, not both normal, that have more than two (A, B)-double cosets. Partition G by the double cosets, and highlight these on the Cayley graph by coloring the nodes. Also partition G by the left and right cosets of AB, if this is a subgroup. Please use an example that is different from all of your homework collaborators.

3. The *centralizer* of an element $h \in G$ is the set of elements that commute with it:

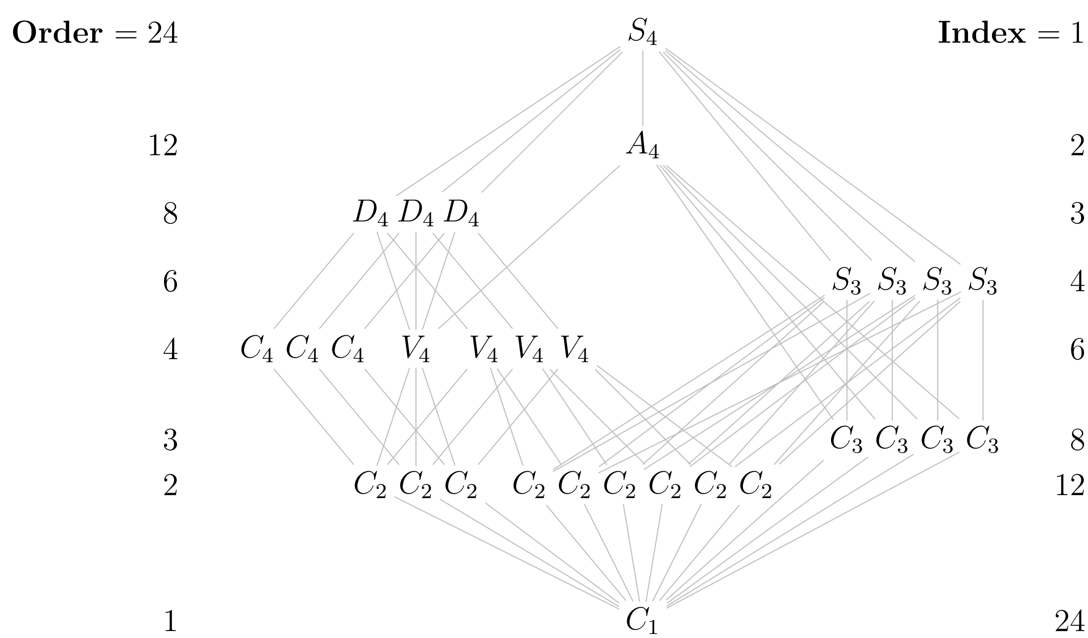
$$C_G(h) := \{g \in G \mid gh = hg\}.$$

   (a) Find the centralizers of the elements r, s, and r^3 in both groups from Problem 1.
   (b) Show that $C_G(H) \trianglelefteq N_G(H)$.
   (c) Let $h \in G$ with $[G : \langle h \rangle] = n < \infty$. Show that there are exactly $[G : C_G(h)]$ elements conjugate to h, by constructing a map from the right cosets of $C_G(h)$ to the conjugacy class $\mathrm{cl}_G(h)$ of h, and showing that it is a well-defined bijection.
   (d) Partition both groups from Part (a) by conjugacy classes.

4. Let G be a group, not necessarily finite, and $H \leq G$ a subgroup.

   (a) Show that the the subgroup $N := \bigcap_{x \in G} xHx^{-1}$ is normal in G.
   (b) Show that every normal subgroup $K \trianglelefteq G$ contained in H is contained in N. In other words, N is the *largest normal subgroup* of G contained in H.
   (c) Show that if $[G : N] < \infty$ and $|H| = \infty$, then $|H \cap N| > 1$.

5. Answer the following questions about the symmetric and alternating groups. The subgroup lattice of $G = S_4$ is shown below.



   (a) Partition the subgroups of S_4 into conjugacy classes, and justify your answer. You can assume that two elements are conjugate iff they have the same cycle type.
   (b) For each $\mathrm{cl}_{S_4}(H)$, find the isomorphism type of the normalizer, $N_{S_4}(H)$.
   (c) Compute the centralizers of e, (12), (123), (1234), and $(12)(34)$ in S_4.
   (d) Partition the elements of A_4 by conjugacy class. Then pick one element σ from each class, and find its centralizer, $C_{A_4}(\sigma)$.
   (e) For each of the following elements $\sigma \in S_5$, find $|\,\mathrm{cl}_{S_5}(\sigma)|$, and then its centralizer $C_{S_5}(\sigma)$: e, (12), (123), (1234), (12345), $(12)(34)$, and $(123)(45)$ in S_5.