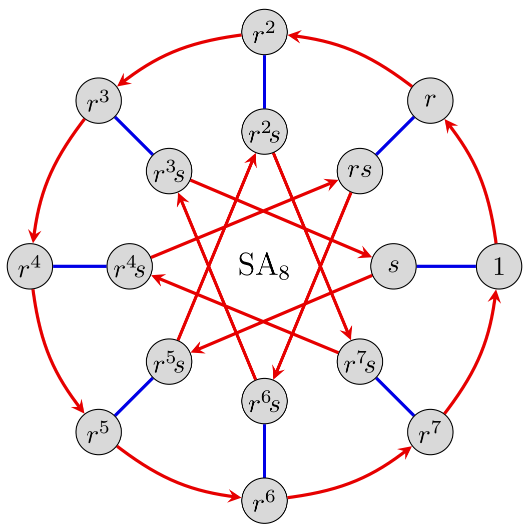
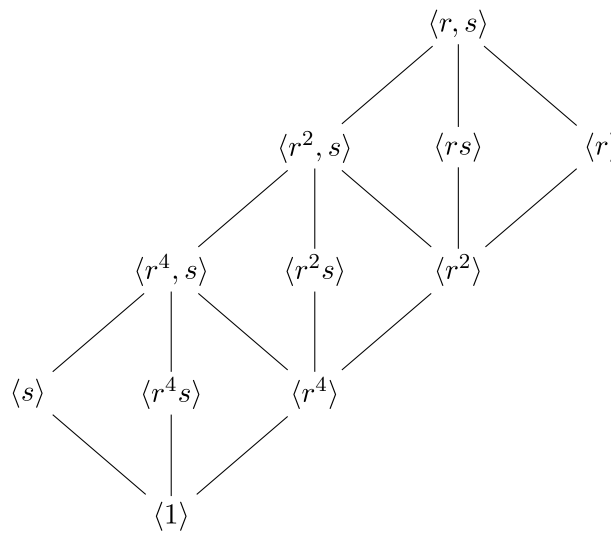
1. Establish the following isomorphisms by defining an explict map and proving that it is a bijective homomorphism.
   (a) $H \cong xHx^{-1}$, for any subgroup $H \leq G$ and fixed $x \in G$.
   (b) $(\mathbb{Q}^*, \cdot\,) \cong (\mathbb{Q}^+, \cdot\,) \times C_2$, where $C_2 = \{1, -1\}$.

2. Let G be the *semiabelian group* of order 16, defined by the presentation

$$\mathrm{SA}_8 = \langle r, s \mid r^8 = s^2 = 1, srs = r^5 \rangle,$$

A Cayley diagram and subgroup lattice are shown below.





   (a) The subgroups $V = \langle r^4, s\rangle$, $H = \langle r^2 s\rangle$, $K = \langle r^2\rangle$, and $N = \langle r^4\rangle$ are all normal. Highlight their cosets on a fresh Cayley diagram by colors.
   (b) Construct a Cayley table for the quotient of G by each of these subgroups. Then draw a Cayley diagram for each, labeling the nodes with elements (i.e., cosets).
   (c) Let $N = \langle r^4\rangle$. The shaded region below shows an order-4 cyclic subgroup of G/N, generated by the element rN, and how the union of these four cosets is the order-8 subgroup $\langle r\rangle$ of G. Construct analogous tables for the other five non-trivial proper subgroups of G/N, and then draw the subgroup lattice of G/N.

r^3N	r^3sN
r^2N	r^2sN
rN	rsN
N	sN

$\langle rN\rangle \leq G/N$

r^3	r^7	r^3s	r^7s
r^2	r^6	r^2s	r^6s
r	r^5	rs	r^5s
1	r^4	s	r^4s

$\langle r\rangle/N \leq G/N$

r^3	r^7	r^3s	r^7s
r^2	r^6	r^2s	r^6s
r	r^5	rs	r^5s
1	r^4	s	r^4s

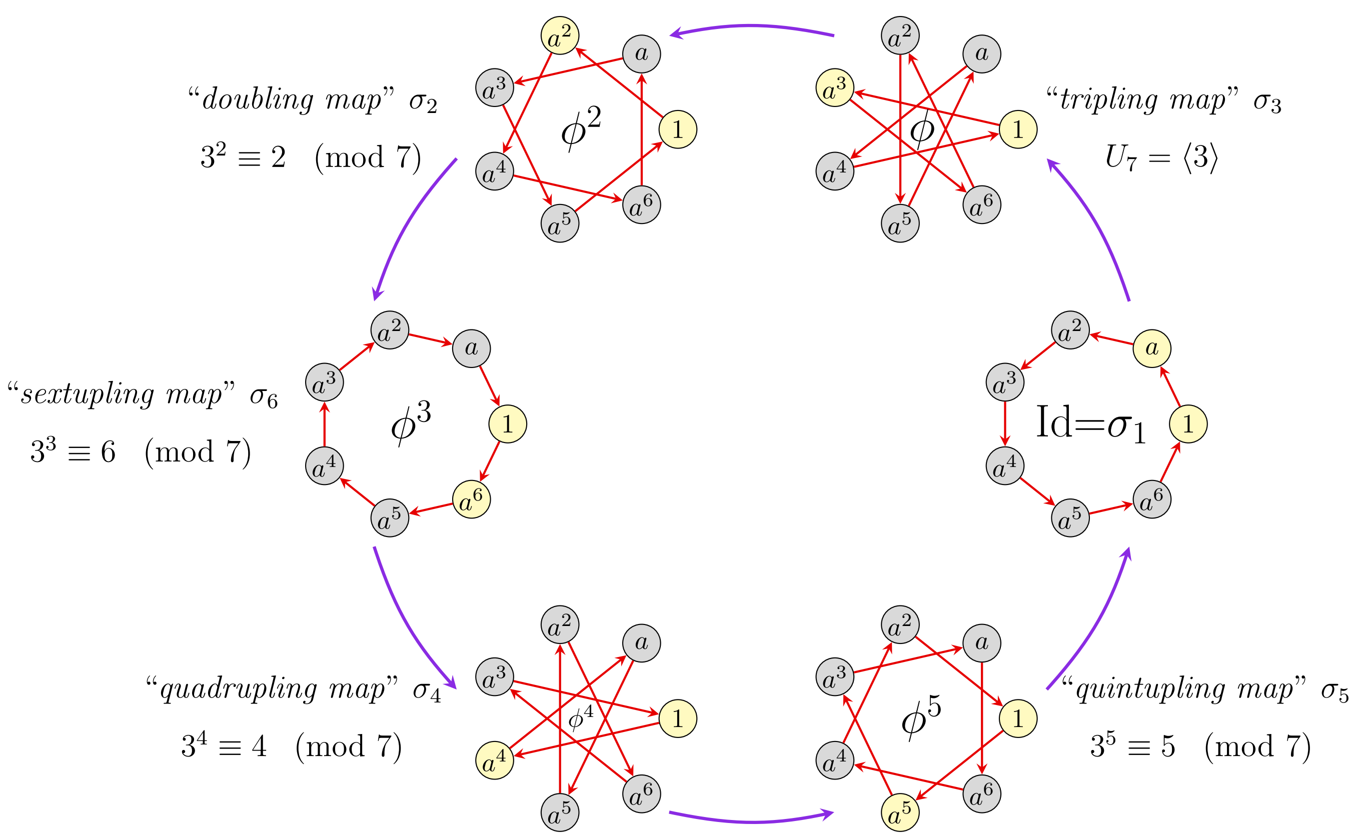
$\langle r\rangle \leq G$

   (d) Repeat the previous part for the subgroups H, K, and V, but include the trivial and nonproper subgroups.

3. Suppose $A, B \leq G$, and that A normalizes B. That is, $A \leq N_G(B)$.
   (a) Show that $AB \leq G$.
   (b) Show that $B \trianglelefteq AB$ and $A \cap B \trianglelefteq A$.
   (c) Show that $A/(A \cap B) \cong AB/B$.
   (d) Show that if A and B are both normal with $G = AB$, then
   $$G/(A \cap B) \cong (G/A) \times (G/B).$$

4. Prove the remaining parts of the correspondence theorem. That is, if $N \leq H \leq G$ is a chain of subgroups and $N \trianglelefteq G$, then show all of the following.
   (a) $H/N \trianglelefteq G/N$ if and only if $H \trianglelefteq G$
   (b) $[G/N : H/N] = [G : H]$
   (c) $H/N \cap K/N = (H \cap K)/N$
   (d) $\langle H/N, K/N \rangle = \langle H, K \rangle / N$
   (e) H/N is conjugate to K/N in G/N if and only if H is conjugate to K in G.

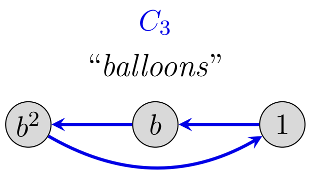
5. The semidirect product $C_7 \rtimes C_3$ can be constructed in a "visual" manner by starting with a Cayley graph of $\mathrm{Aut}(C_7) \cong U_6 \cong C_6$, shown below, with the nodes labeled by "rewirings" of a Cayley graph of C_7.

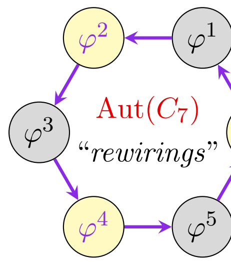


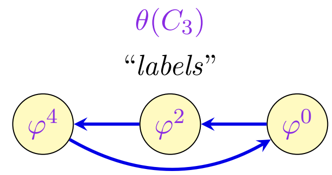
Next, we define the "labeling map," a homomorphism

$$\theta\colon C_3 \longrightarrow \mathrm{Aut}(C_7), \qquad \theta\colon b^k \longmapsto \phi^{2k},$$

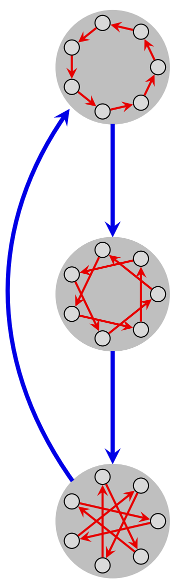
which tells us how to "stick in" rewired copies of $A = C_7$ into "inflated" nodes of $B = C_3$.

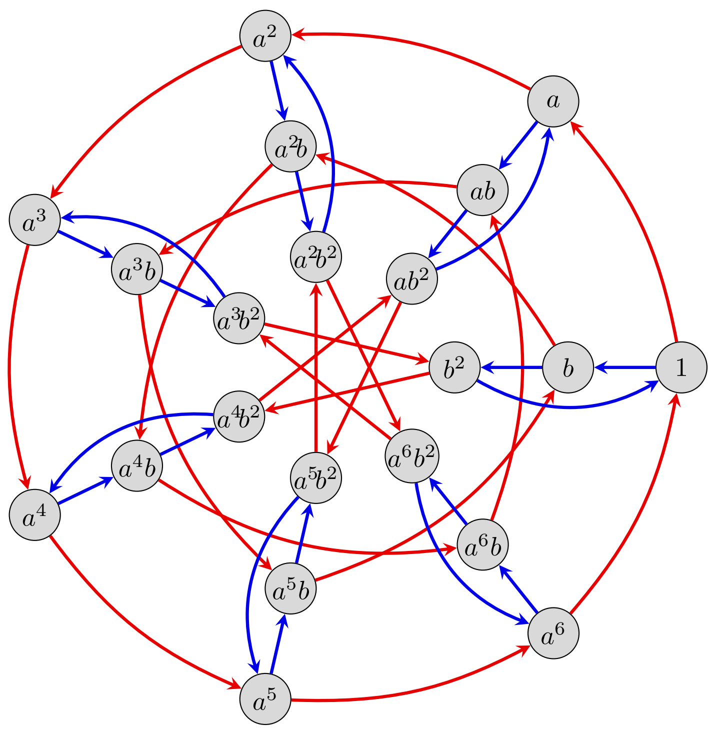






By connecting up the corresponding nodes, we get a Cayley graph for $C_7 \rtimes C_3$, like the following.





(a) Carry out analogous steps to construct Cayley graphs of $C_9 \rtimes C_3$ and $C_3 \rtimes C_6$, and then write down a presentation for each group. [*Hint*: There are two homomorphisms $C_3 \to \mathrm{Aut}(C_9)$ that will work, but one of them leads to a much less tangled diagram.]

(b) In each case, this group is isomorphic to the Cartesian product $G = A \times B$ with a different binary operation than the direct product. Give an explicit formula for $(a^i, b^j) * (a^k, b^\ell)$ using this new binary operation.