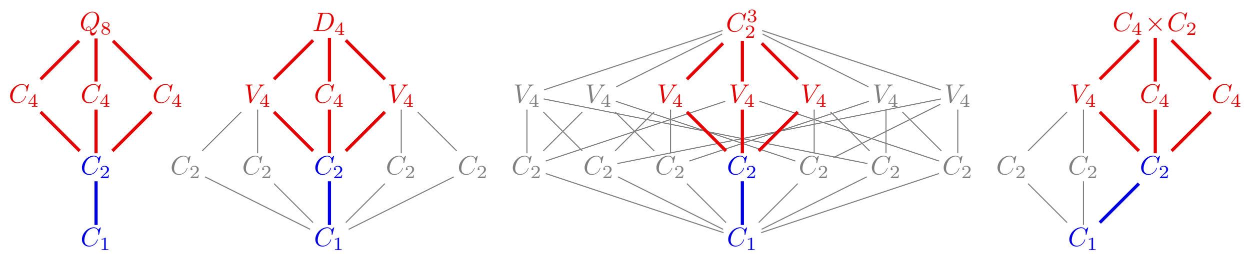
1. The *short five lemma* says that given the commutative diagram of exact sequences shown at left, if ν and κ are isomorphisms, then γ is as well.

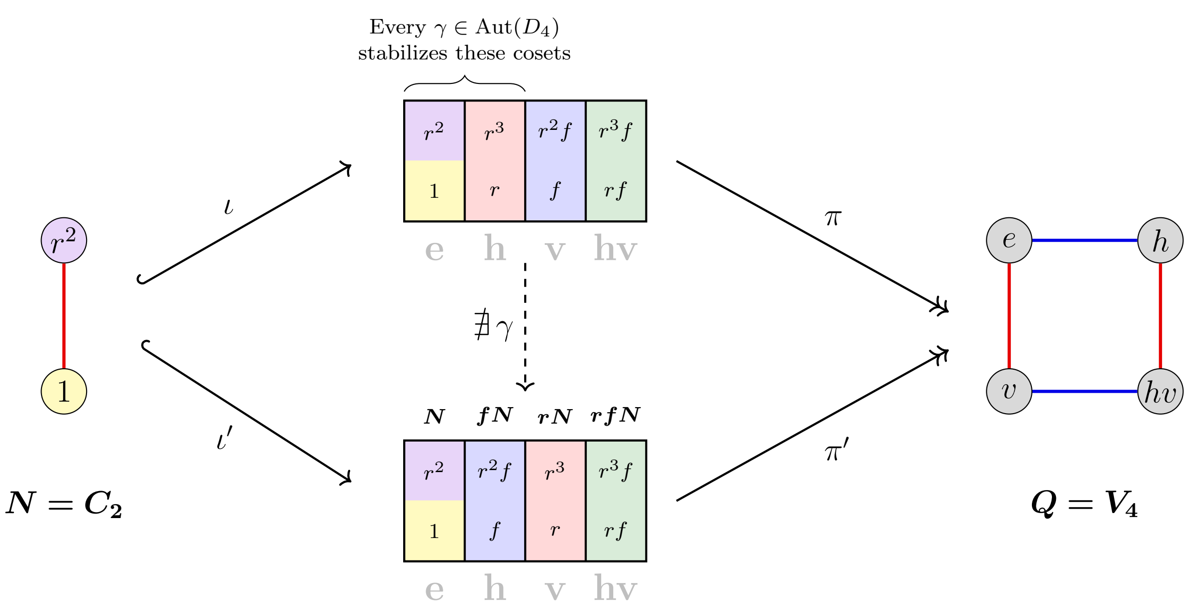
$$\begin{tikzcd}
1 \arrow[r] & N_1 \arrow[r, "\iota_1"] \arrow[d, "\nu"] & G_1 \arrow[r, "\pi_1"] \arrow[d, "\gamma"] & Q_1 \arrow[r] \arrow[d, "\kappa"] & 1 \\
1 \arrow[r] & N_2 \arrow[r, "\iota_2"] & G_2 \arrow[r, "\pi_2"] & Q_2 \arrow[r] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & G \arrow[dr, two heads, "\pi"] \arrow[dd, dashed, "\gamma"] & & \\
1 \arrow[r] & N \arrow[ur, hook, "\iota"] \arrow[dr, hook, "\iota'"'] & & Q \arrow[r] & 1 \\
 & & G \arrow[ur, two heads, "\pi'"'] & &
\end{tikzcd}$$

   (a) Prove the short five lemma.

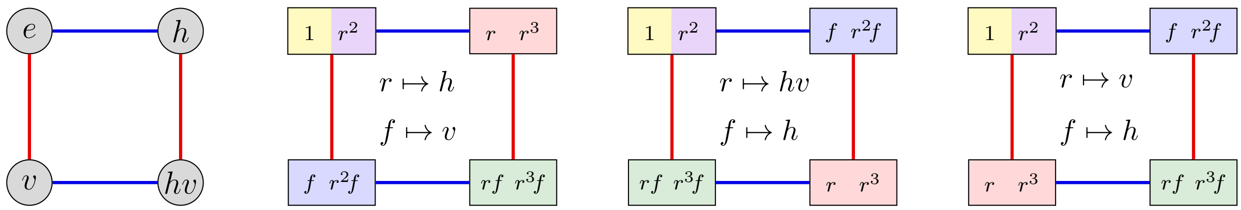
   (b) Two extentions $N_i \overset{\iota_i}{\hookrightarrow} G_i \overset{\pi_i}{\twoheadrightarrow} Q_i$ are said to be *equivalent* if they are related via a commutative diagram, like the one above (left). Two extensions of Q by N are equivalent if they are related by a commutative diagram like the one on the right. Up to *isomorphism*, there are four extensions of $Q = V_4$ by $N = C_2$, as shown below.



However, up to equivalence, there are more than four. For example, finding all extesnions $C_2 \overset{\iota}{\hookrightarrow} D_4 \overset{\pi}{\twoheadrightarrow} V_4$ amounts to finding all $\gamma \in \mathrm{Aut}(D_4)$ that makes the diagram commute. Since each γ fixes the cosets $\{1, r^2\}$ and $\{r, r^3\}$, the following diagram illustrates two quotients that cannot be equivalent.



Each of the three choices of the image $\pi(r) \in \{h, v, hv\}$ characterizes an extension $C_2 \overset{\iota}{\hookrightarrow} D_4 \overset{\pi}{\twoheadrightarrow} V_4$. Examples of these are shown below.

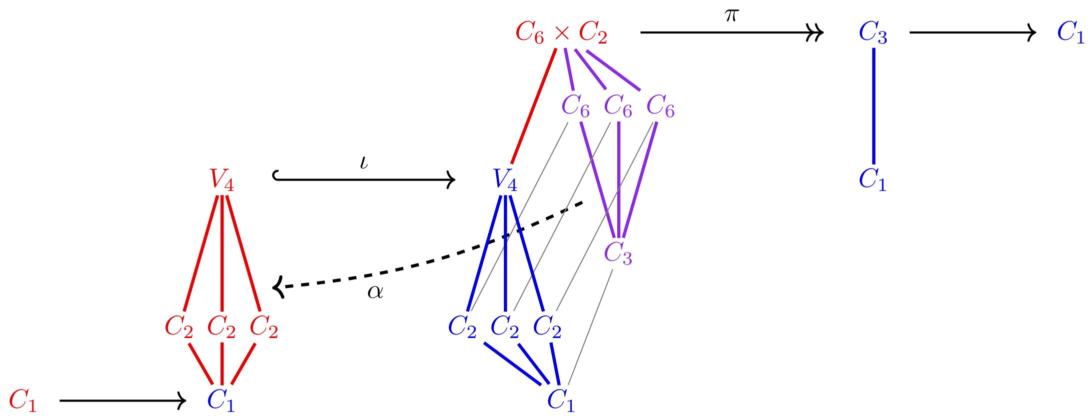


Carry out the previous steps, including the visuals, to classify the extensions $C_2 \overset{\iota}{\hookrightarrow} C_4 \times C_2 \overset{\pi}{\twoheadrightarrow} V_4$, and then do the same for the extensions $C_2 \overset{\iota}{\hookrightarrow} Q_8 \overset{\pi}{\twoheadrightarrow} V_4$.

2. A short exact sequence of groups is *left split* if there is a "backwards map" $\alpha \colon G \to N$ for which $\alpha \circ \iota = \mathrm{Id}_N$, like the following shows.

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r, "\iota"] & G \arrow[l, bend left, dashed, "\alpha"] \arrow[r, "\pi"] & H \arrow[r] & 1 \end{tikzcd}$$

A picture of a left split exact sequence $1 \to V_4 \to C_6 \times C_2 \to C_3 \to 1$ is shown below.



(a) Show that every split exact sequence of abelian groups is left split.

(b) Show that if a short exact sequence is left split, then it is (right) split. Must G (the middle term) be abelian in this case? Justify your answer.

(c) Give an example of a right split exact sequence that is not left split.

3. Let $C := \{aba^{-1}b^{-1} \mid a, b \in G\}$ be the set of commutators of a group G. Prove the following facts about the *commutator subgroup*, $G' = \langle C \rangle$.

(a) $G' = \displaystyle\bigcap_{C \subseteq N \trianglelefteq G} N$.

(b) $G' \trianglelefteq G$ and G/G' is abelian.

(c) If $G' \leq H \leq G$, then $H \trianglelefteq G$.

(d) If $N \trianglelefteq G$, then $N' \trianglelefteq G$.

(e) If $\phi \colon G \twoheadrightarrow H$, then H is abelian if and only if $G' \leq \mathrm{Ker}(\phi)$.

4. Prove the following, where G is a finite group, p a prime divisor of $|G|$, and

$$G'(p) := \bigcap \big\{ N \trianglelefteq G \mid G/N \text{ is an abelian } p\text{-group} \big\}.$$

   (a) $G/G'(p)$ is an abelian p-group.
   (b) $G/G'(p)$ is isomorphic to the unique Sylow p-subgroup of G/G'.
   (c) $p \nmid [G'(p) : G']$.
   (d) If P is any Sylow p-subgroup of G, then $PG'(p) = G$.

5. For both of the following groups shown below, find all compositions series (up to isomorphism), the composition factors, the derived series, and abelianization.