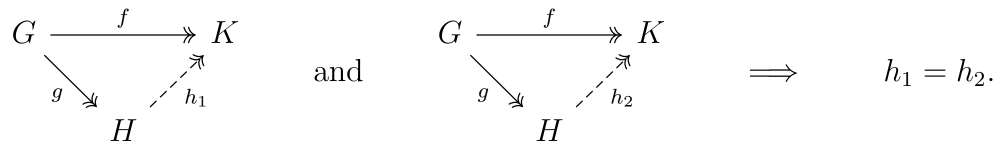
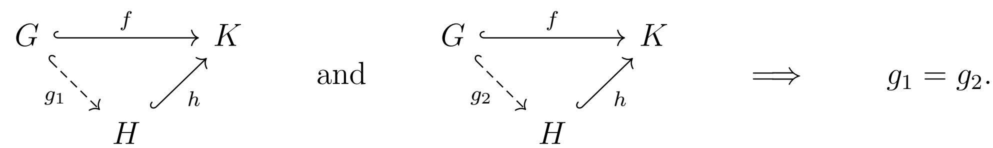
1. Consider homomorphisms $g_i \colon G \to H$ and $h_i \colon H \to K$ between groups, for $i = 1, 2$.

   (a) Show that if $g$ is surjective, then it right-cancels: $h_1 \circ g = h_2 \circ g \implies h_1 = h_2$.



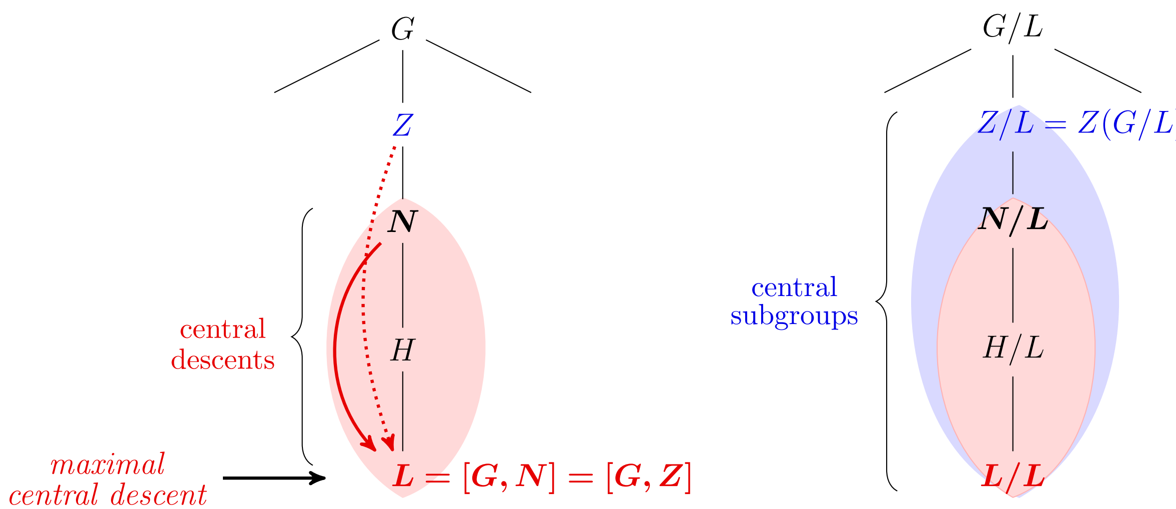

   (b) Show that if $h$ is injective, then it left-cancels: $h \circ g_1 = h \circ g_2 \implies g_1 = g_2$.




   (c) Give explicit examples to show how both of the previous results can fail if the hypotheses are not met.

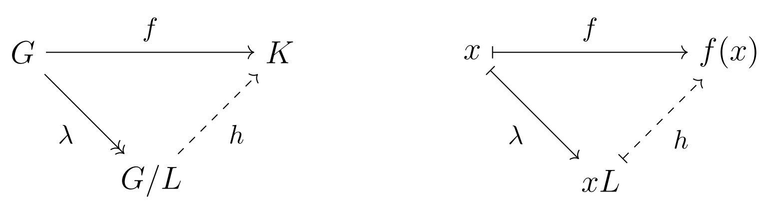
2. We defined the ascending central series via iterative "maximal central descents." Given $N \trianglelefteq G$, the *maximal central descent* $[G, N]$ is characterized as being

   "*the smallest subgroup $L$ such that $N/L$ is central in $G/L$.*"

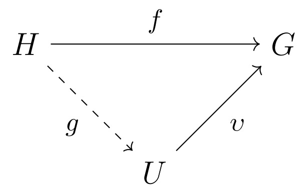



   Prove that $(L, \lambda)$ satisfies the following co-universal property, where $L = [G, N]$ and $\lambda \colon G \twoheadrightarrow G/L$ is the canonical quotient.

   "*If $N \trianglelefteq G$ and $f \colon G \to K$ for which $f(N)$ is central, then $f$ uniquely factors through the canonical quotient map $\lambda \colon G \to G/L$, where $L = [G, N]$. That is, there is a unique homomorphism $h \colon G/L \to K$ for which $f = h \circ \lambda$.*"



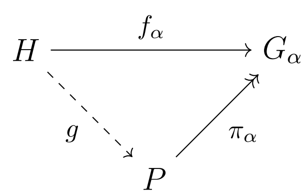
3. A *universal pair* $(U, \upsilon)$ for a group $G$ with respect to a particular property consists of a group $U$ with an outgoing map $\upsilon\colon U \to G$, such that every other map $f\colon H \to G$ with that same property factors through $\upsilon$ uniquely. That is, there is a unique homomorphism $g\colon H \to U$ between the *domains* such that $f = \upsilon \circ g$.




Prove that if $G$ has a universal pair $(U, \upsilon)$ with with respect to some property, then $U$ is unique up to isomorphism.

4. The *product* of $\{G_\alpha \mid \alpha \in I\}$ is a group $P$ with a family of homomorphisms $\{\pi_\alpha\colon P \to G_\alpha \mid \alpha \in I\}$, satisfying the following universal property:

> "*Given any group $H$ and homomorphisms $f_\alpha\colon H \to G_\alpha$, there is a unique homomorphism $g\colon H \to P$ such that $\pi_\alpha \circ g = f_\alpha$ for all $\alpha \in I$.*"

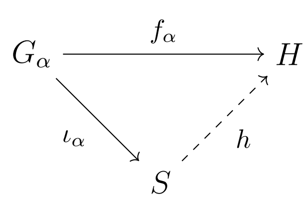



   (a) Show that $Z(\prod_\alpha G_\alpha) = \prod_\alpha Z(G_\alpha)$.

   (b) Show that $(G_1 \times \cdots \times G_n)' = G_1' \times \cdots \times G_n'$.

5. The *co-product* of $\{G_\alpha \mid \alpha \in I\}$ is a group $S$ with a family of homomorphisms $\{\iota_\alpha\colon G_\alpha \to S \mid \alpha \in I\}$, satisfying the following co-universal property:

> "*Given any group $H$ and homomorphisms $f_\alpha\colon G_\alpha \to H$, there is a unique homomorphism $h\colon S \to H$ such that $h \circ \iota_\alpha = f_\alpha$ for all $\alpha \in I$.*"




   (a) Show that if a non-empty family of groups $\{G_\alpha \mid \alpha \in I\}$ has a co-product, then it is unique up to isomorphism, and each $\iota_\alpha$ is injective.

   (b) Prove that the set of finite sums $\sum_{\alpha \in I} A_\alpha$ is a coproduct in the category of abelian groups, where $\iota_\alpha$ are the canonical injections.