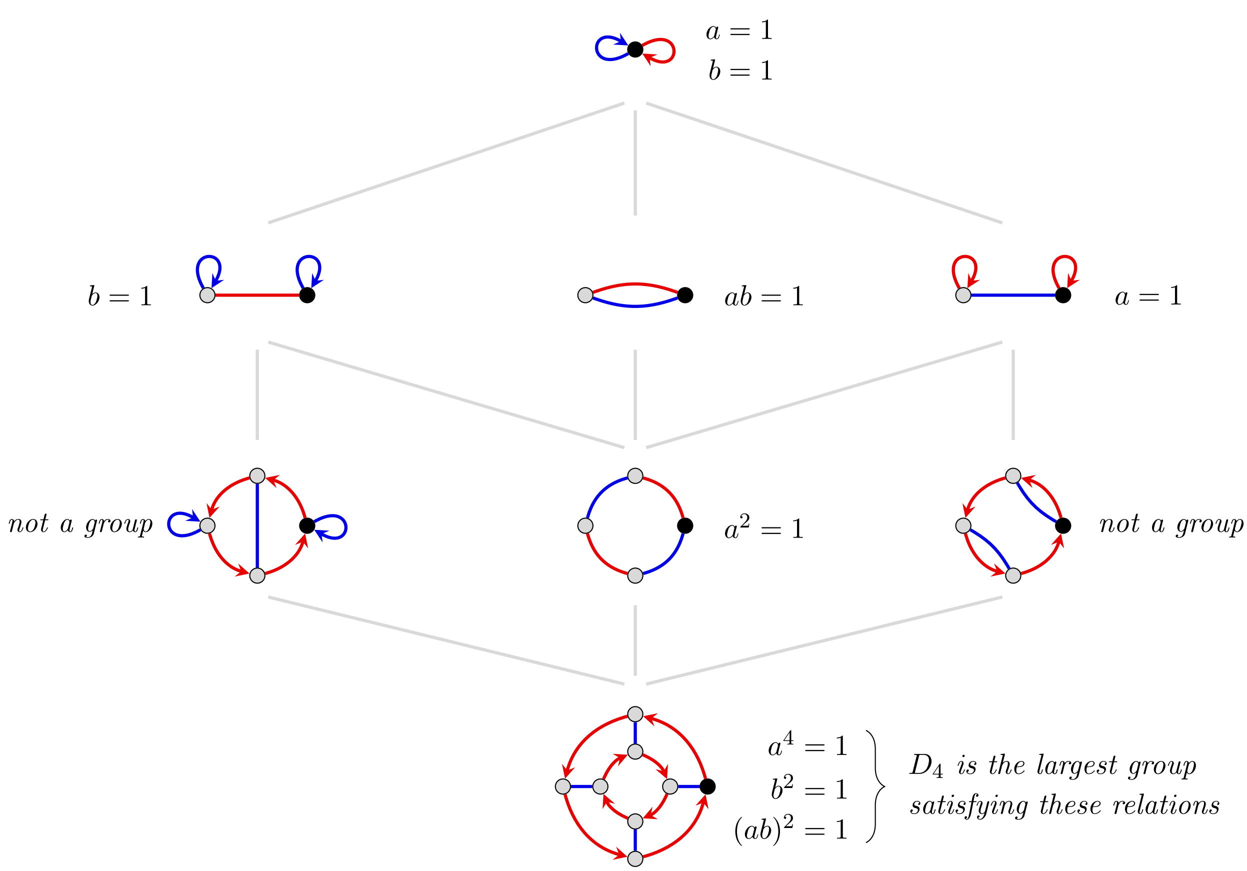
1. Every group $G = \langle a, b \rangle$ satisfying the relations $a^4 = 1$, $b^2 = 1$, and $(ab)^2 = 1$ is isomorphic to a quotient of D_4. Said differently, D_4 is the *largest* group on two generators that satisfies these relations. The figure below shows all of the distinct ways to collapse the Cayley diagram for D_4 by right cosets of a subgroup, and the correponding relation(s) added, if the result is a group.

   (a) Characterize D_4 by a univeral or co-universal property involving relations, and include the correct commutative diagram.

   (b) Create an analogous figure for $D_6 = \langle a, b \mid a^6 = b^2 = (ab)^2 = 1 \rangle$. You should already have the completed conjugacy poset from HW 4.

   (c) Create an analogous figure for $D_6 = \langle s, t \mid s^2 = t^2 = (st)^6 = 1 \rangle$.



2. Let $\mathcal{C}$ be a category.

   (a) If $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is an isomorphism, show that there is at most one $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$.

   (b) Prove that any two terminal objects in $\mathcal{C}$ are equivalent.

   (c) Suppose a family of objects $\{A_i \mid i \in I\}$ has a coproduct S, with morphisms $\iota_j \in \mathrm{Hom}(A_j, S)$. Show that if $\mathcal{C}$ has a zero object, then each ι_j is a monomorphism.

3. Let (F, ι) be a free group on a set S.
   (a) Show that $F = \langle \iota(S) \rangle$.
   (b) If $|S| \geq 2$, show that F is not abelian.
   (c) For any $T \subseteq S$, show that there exists a normal subgroup $N \trianglelefteq F$ such that $(F/N, \pi\iota|_T)$ is a free group on T.
   (d) Show that $(\langle \iota(T) \rangle, \iota|_T)$ is a free group on T.

4. Let F be a free object on $S \neq \emptyset$ in a concrete category $\mathcal{C}$. Prove that if F' is a free object on a set S' with $|S| = |S'|$, then F and F' are equivalent.

5. For a positive integer n, let **Nil** be the category of nilpotent groups, and let $\mathbf{Nil}_{\leq n}$ be the category of nilpotent groups of class at most n.
   (a) Show that there cannot exist a nilpotent group N generated by two elements with the property that every nilpotent group generated by two elements is a homomorphic image of N.
   (b) Prove or disprove that free objects always exist in **Nil**.
   (c) Prove or disprove that free objects always exist in $\mathbf{Nil}_{\leq n}$.