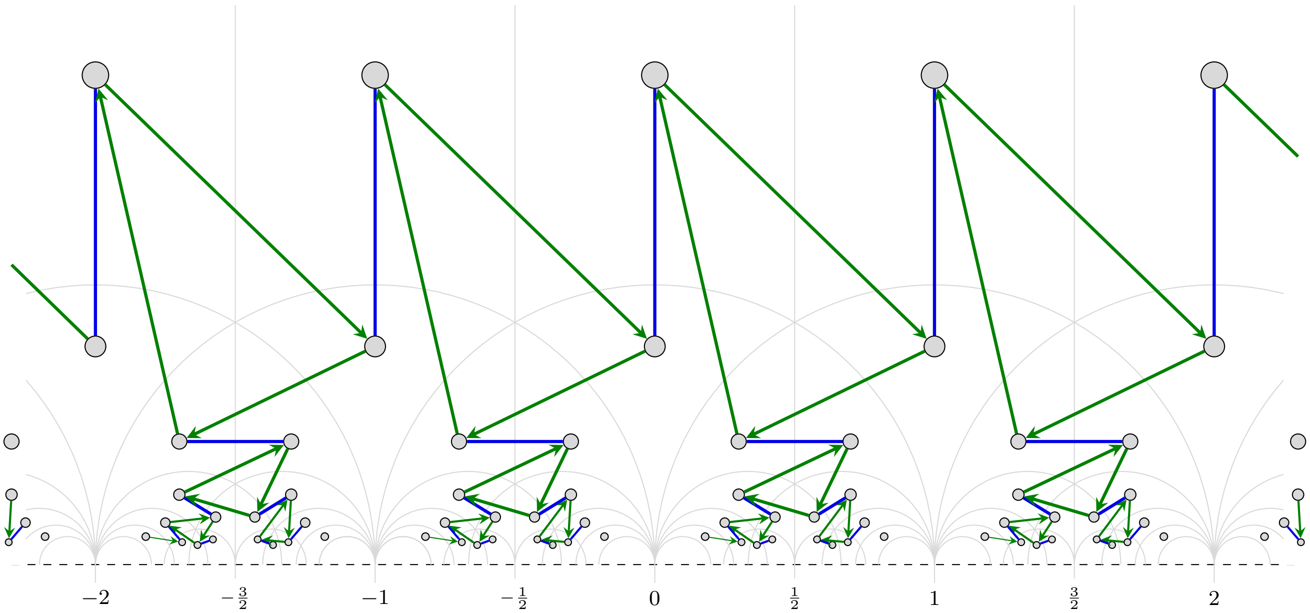
1. A Cayley graph of the *projective linear group* $\mathrm{PSL}_2(\mathbb{Z}) = \langle A, B\rangle$ is shown below.



The generators are the images $A = \pi(ST)$ and $B = \pi(S)$ under the natural quotient map $\pi\colon \mathrm{SL}_2(\mathbb{Z}) \twoheadrightarrow \mathrm{PSL}_2(\mathbb{Z})$, where

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad ST = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

Thus, we can think of them as $A = \pm S$ and $B = \pm ST$. In this problem, we will show that the identity element in $\mathrm{PSL}_2(\mathbb{Z})$ cannot be written nontrivially as

$$I = A^{i_1} B^{j_1} A^{i_2} B^{j_2} \cdots A^{i_{m-1}} B^{j_{m-1}} A^{i_m}, \qquad i_k \in \{0,1,2\}, \quad j_k \in \{0,1\}.$$

This will confirm that $\mathrm{PSL}_2(\mathbb{Z}) \cong C_3 * C_2$, which is suggested by the Cayley graph above.

(a) Show by slight brute force that this is impossible for $m = 1$ and $m = 2$.

(b) Now, suppose there is such a representation of the idenitity, for some $m \geq 3$. Assuming that m is minimal, left-multiply by A^{-i_1} and right-multiply by A^{i_1}. Show that $i_m + i_0$ is not a multiple of 3, and conclude that the identity element can be written as a product of BA's and BA^2's.

(c) Recalling that $A = \pm ST$ and $B = \pm S$, let $R = ST$, and consider the following matrices in $\mathrm{SL}_2(\mathbb{Z})$:

$$SR = S^2 T = -T = -\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad SR^2 = -TST = -\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

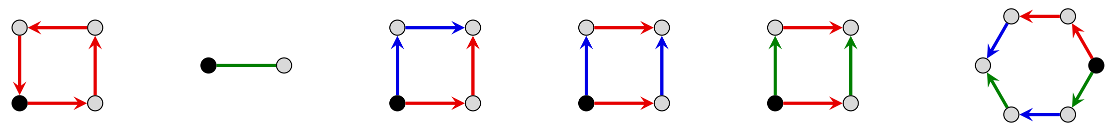
Show that for any (nontrivial) product of these matrices, the absolute sum of the entries is at least 3.

(d) Conclude that the identity element in $\mathrm{PSL}_2(\mathbb{Z})$ cannot be written nontrivially.

2. Consider the "mystery group" $M = \langle S_1 \mid R_1 \rangle$ defined by the following presentation.

$$M = \big\langle a, b, c \mid a^4,\ c^2,\ a^2b^{-2},\ aba^{-1}b^{-1},\ aca^{-1}c,\ a^2bc^{-1}b^{-1}c^{-1} \big\rangle.$$

The relators of this presentations describe the following motifs that a Cayley graph for $M = \langle a, b, c \rangle$ must have.



$a^4 = 1$ $\quad$ $c^2 = 1$ $\quad$ $a^2 = b^2$ $\quad$ $ab = ba$ $\quad$ $ac = ca$ $\quad$ $a^2b = cbc$

(a) Establish $|M| \leq 16$ by showing that every word in M can be written

$$a^i b^j c^k, \qquad i \in \{0, 1, 2, 3\}, \quad j \in \{0, 1\}, \quad k \in \{0, 1\},$$

(b) Identify a "familiar group" $F = \langle S_2 \mid R_2 \rangle$ of order 16 whose generators satisfy these relations. That is, define a "relabing map" $\theta \colon S_1 \twoheadrightarrow S_2$ that extends to $\theta \colon R_1 \to R_2$.

(c) Describe why it follows that $M \cong F$.

3. Prove what group is described by each presentation.

(a) $G = \langle a, b \mid a^2 = 1,\ b^3 = 1,\ ab = ba \rangle$

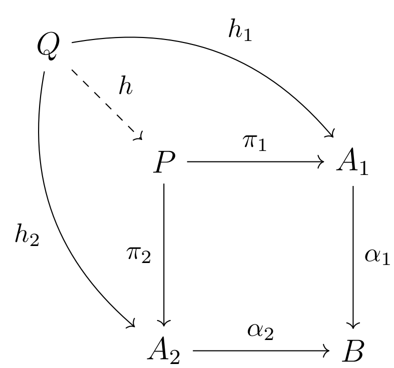
(b) $G = \langle a, b \mid a^4 = 1,\ a^2 = b^2,\ ab = ba^3 \rangle$

(c) $G = \langle a, b \mid a^4 = b^3 = 1,\ ab = ba^3 \rangle$

(d) $G = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (ac)^2 = (bc)^3 = 1 \rangle$.

4. If A_1, A_2, and B are objects in a category $\mathcal{C}$ with morphisms $\alpha_i \in \mathrm{Hom}_{\mathcal{C}}(A_i, B)$, then their *fiber product* (or *pullback*) is an object P with morphisms $\pi_i \in \mathrm{Hom}_{\mathcal{C}}(P, A_i)$ such that the following property holds:

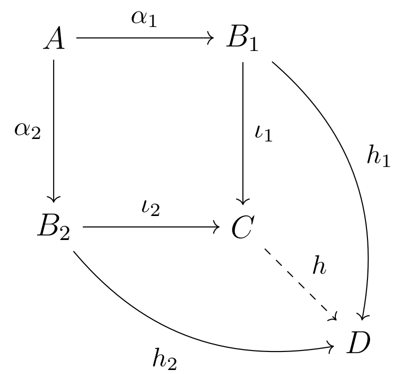
> "*For any object Q in $\mathcal{C}$ and morphisms $h_i \in \mathrm{Hom}_{\mathcal{C}}(Q, A_i)$ such that if $\pi_1 \circ h_1 = \pi_2 \circ h_2$, there exists a unique morphism $h \in \mathrm{Hom}_{\mathcal{C}}(Q, P)$ such that $h_i = \pi_i \circ h$ for $i = 1, 2$.*"



Prove that any two pullbacks are equivalent.

5. Let A, B_1, B_2 be objects in a category $\mathcal{C}$ and let $\alpha_i \in \mathrm{Hom}_{\mathcal{C}}(A, B_i)$ for $i = 1, 2$. A *fiber coproduct* (or *pushout*) is an object C with morphisms $\iota_i \in \mathrm{Hom}(B_i, C)$ satisfying the following couniversal property:

   For any object $D \in \mathrm{Ob}(\mathcal{C})$ and morphisms $h_i \in \mathrm{Hom}_{\mathcal{C}}(B_i, D)$ such that if $h_1 \circ \alpha_1 = h_2 \circ \alpha_2$, there exists a unique $h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ such that $h \circ \iota_i = h_i$.



   Let Y and Z be sets with inclusion maps $\alpha_Y \colon Y \cap Z \hookrightarrow Y$ and $\alpha_Z \colon Y \cap Z \hookrightarrow Z$. Show that the pushout (or fiber coproduct) of α_Y and α_Z is equivalent to the union $Y \cup Z$, as illustrated by the following commutative diagram.