1. Let I and J be ideals of a commutative ring R.

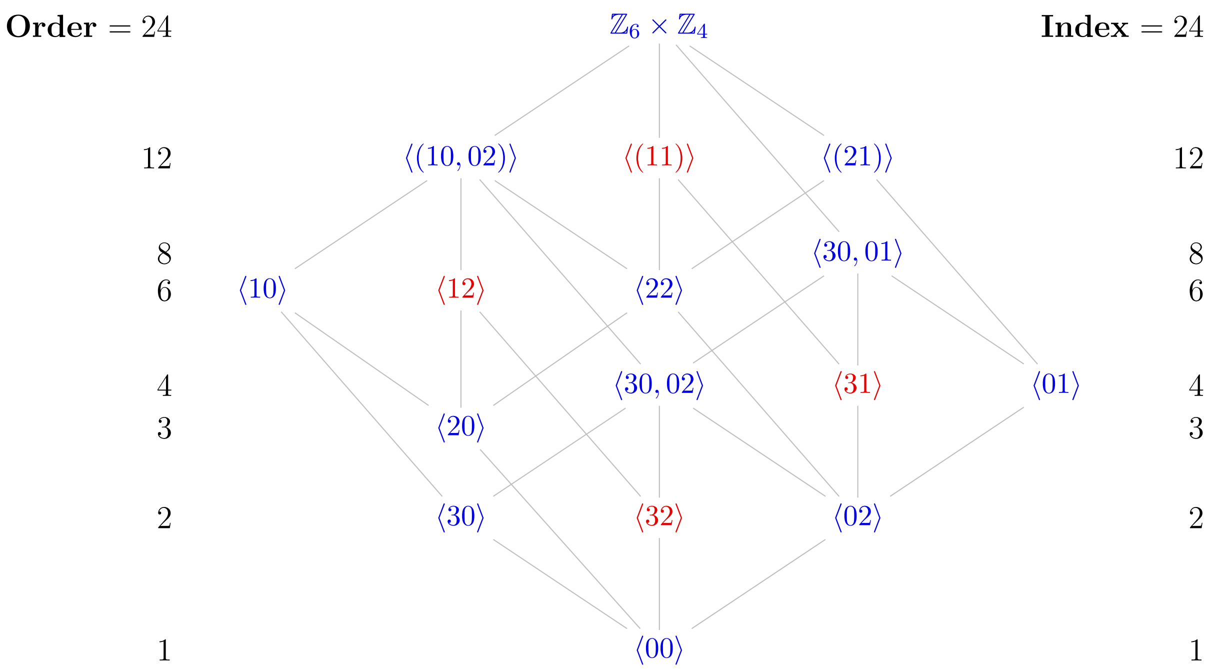
   (a) Show that $I+J$, $I\cap J$, and IJ are ideals of R. Which of these remain ideals if the commutativity hypothesis is dropped?

   (b) The set $(I:J) := \{r \in R \mid rJ \subseteq I\}$ is called the *ideal quotient* or *colon ideal* of I and J. Show that $(I:J)$ is an ideal of R. Does this require commutativity?

   (c) Determine $I+J$, $I\cap J$, IJ, and $(I:J)$ for the ideals $I=n\mathbb{Z}$ and $J=m\mathbb{Z}$ of $R=\mathbb{Z}$.

   (d) Repeat Part (c) for several pairs of ideals of $R=\mathbb{Z}_6\times\mathbb{Z}_4$, whose subring lattice is shown below.

   (e) Describe how to find IJ and $(I:J)$ by inspection, using only the subring lattice, if possible.



2. Let $f: R \to S$ be a ring homomorphism between commutative rings.

   (a) If f is surjective and I is an ideal of R, show that $f(I)$ is an ideal of S.

   (b) Show that Part (a) is not true in general when f is not surjective.

   (c) Show that if f is surjective and R is a field, then S is a field as well.

3. Let R be a commutative ring.

   (a) Show that if x is contained in every maximal ideal, then $1+x$ is a unit.

   (b) A ring is *local* if it has a unique maximal ideal. Show that R is local if and only if the non-units form an ideal.

   (c) Characterize units and maximal ideals of the ring

$$R = \left\{ \tfrac{a}{b} \mid a,b \in \mathbb{Z},\ (a,b)=1,\ p \nmid b \right\} \subseteq \mathbb{Q},$$

   where p is a fixed prime.

4. Let R be a commutative ring. In this problem, we will define several different "radicals" of an ideal I.

   (a) The *radical* of $I \subseteq R$ is the set

   $$\sqrt{I} := \{x \in R \mid x^n \in I \text{ for some } n \in \mathbb{N}\},$$

   and I is a *radical ideal* if $\sqrt{I} = I$.

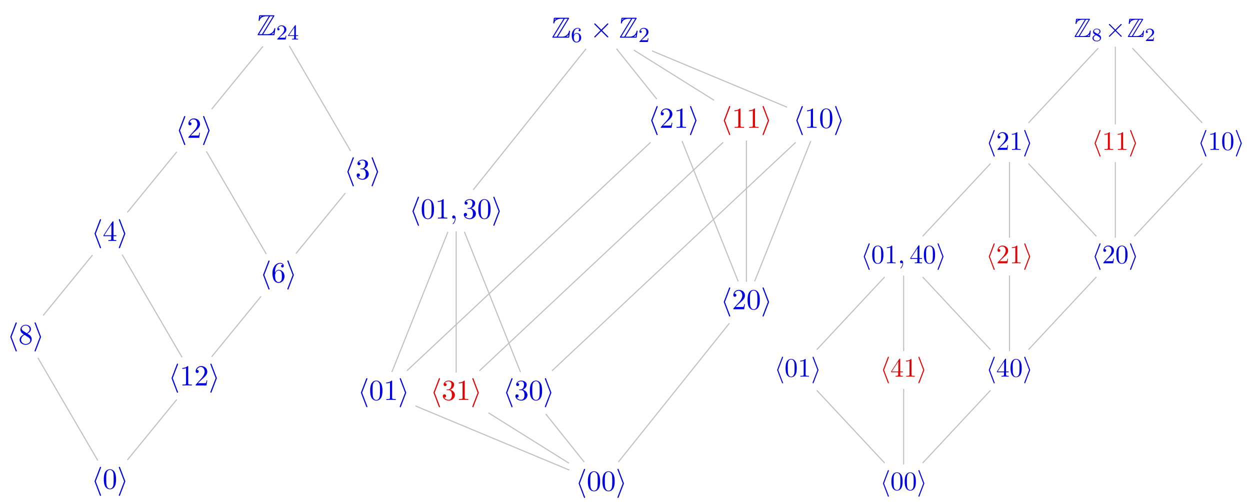
   (i) Show that $\sqrt{I}$ is an ideal containing I.

   (ii) Find the radicals of all ideals of the rings $\mathbb{Z}_6 \times \mathbb{Z}_4$, $\mathbb{Z}_8 \times \mathbb{Z}_2$, $\mathbb{Z}_6 \times \mathbb{Z}_2$, and $\mathbb{Z}_{24}$. Denote these on the subring lattices by drawing an arrow from each I to $\sqrt{I}$.

   (b) The *Jacobsen radical* of I, denoted $J(I)$, is the intersection of all maximal ideals that contain I.

   (i) Show that $J(I)$ is an ideal.

   (ii) Find the Jacobsen radical of all proper ideals of the rings $\mathbb{Z}_{24}$, $\mathbb{Z}_6 \times \mathbb{Z}_4$, $\mathbb{Z}_6 \times \mathbb{Z}_2$, and $\mathbb{Z}_8 \times \mathbb{Z}_2$. Denote these by drawing an arrow from I to $J(I)$ on a fresh copy of the lattices.



5. Let R be commutative. Loosely speaking, a *radical* of R is an ideal of "bad elements."

   (a) An element $a \in R$ is *nilpotent* if $a^n = 0$ for some $n \geq 1$. The *nilradical* of R is $\mathfrak{N}_R := \sqrt{0}$, the set of nilpotent elements.

   (i) If $u \in R$ is a unit and $a \in R$ is nilpotent, show that $u + a$ is a unit.

   (ii) Show that $R/\mathfrak{N}_R$ has no nonzero nilpotent elements.

   (iii) Show that $\mathfrak{N}_{R/I} = \sqrt{I}/I$.

   (b) The *Jacobsen radical of* R is $J(R) := J(0)$, the intersection of all maximal ideals. Show that $J(R) \supseteq \mathfrak{N}_R$.

   (c) Find the Jacobsen and nilradicals of the rings $\mathbb{Z}_{24}$, $\mathbb{Z}_6 \times \mathbb{Z}_4$, $\mathbb{Z}_6 \times \mathbb{Z}_2$, and $\mathbb{Z}_8 \times \mathbb{Z}_2$. Beside each ideal I in the lattice, write $\mathfrak{N}_{R/I}$.