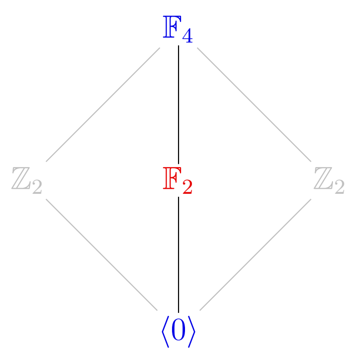
1. Construct the Cayley tables, Cayley graphs, and subring lattices of the finite field $\mathbb{F}_9 \cong \mathbb{Z}_3[x]/(x^2+x+2)$. Examples for the finite fields

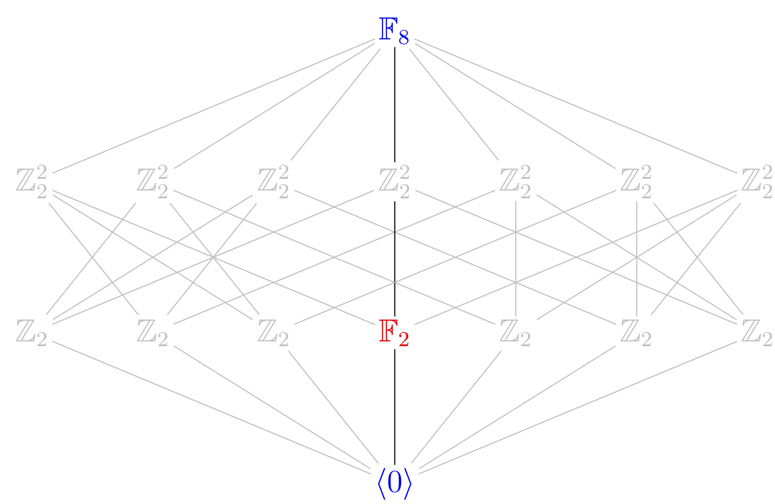
$$\mathbb{F}_4 \cong \mathbb{Z}_2[x]/(x^2+x+1) \qquad \text{and} \qquad \mathbb{F}_8 \cong \mathbb{Z}_2[x]/(x^3+x+1)$$

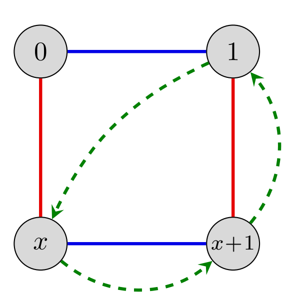
are shown below.

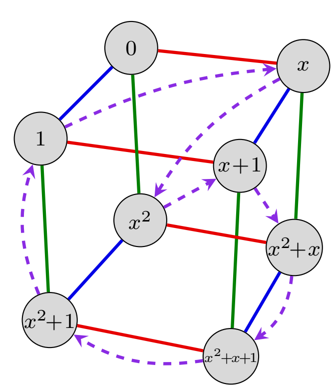
$\times$	1	x	$x+1$
1	1	x	$x+1$
x	x	$x+1$	1
$x+1$	$x+1$	1	x

$\times$	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
1	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
x	x	x^2	x^2+x	$x+1$	1	x^2+x+1	x^2+1
$x+1$	$x+1$	x^2+x	x^2+1	x^2+x+1	x^2	1	x
x^2	x^2	$x+1$	x^2+x+1	x^2+x	x	x^2+1	1
x^2+1	x^2+1	1	x^2	x	x^2+x+1	$x+1$	x^2+x
x^2+x	x^2+x	x^2+x+1	1	x^2+1	$x+1$	x	x^2
x^2+x+1	x^2+x+1	x^2+1	x	1	x^2+x	x^2	$x+1$









2. Use Zorn's lemma to show that the ring $\mathbb{R}$ contains a subring A containing 1 that is maximal with respect to the property that $1/2 \notin A$.

3. Let R be a commutative ring with 1. The *radical* of an ideal $I \subseteq R$ is the set

$$\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some } n \in \mathbb{N}\},$$

and I is a *radical ideal* if $I = \sqrt{I}$. An ideal $I \subsetneq R$ is *primary* if $ab \in I$ implies $a \in I$ or $b^n \in I$ for some $n \in \mathbb{N}$.

   (a) Show that an ideal I is prime if and only if it is primary and radical.
   (b) Characterize the radical of a primary ideal.
   (c) Suppose that R is a PID. Characterize its nonzero primary ideals.
   (d) Show that I is primary if and only if all zero-divisors in R/I are nilpotent.
   (e) Give an example to show that, in general, there are primary ideals which are not prime powers.

4. Let R be commutative ring. For a multiplictive semigroup $D \subseteq R$ containing no zero divisors, let $\mathcal{F} = R \times D$ and define a relation on $\mathcal{F}$ where $(a, b) \sim (c, d)$ if $ad = bc$.

   (a) Show that $\sim$ is an equivalence relation on $\mathcal{F}$.
   (b) Denote the equivalence class of (a, b) by a/b and the set of equivalence classes by $D^{-1}R$ (called the *localization* of R at D). Show that $D^{-1}R$ is a commutative ring with 1.
   (c) If $a \in S$, show that $\{ra/a \mid r \in R\}$ is a subring of $D^{-1}R$ and that $r \mapsto ra/a$ is a monomorphism, therefy identifying R with a subring of $D^{-1}R$.
   (d) Under this identification, show that every $d \in D$ is a unit in $D^{-1}R$.
   (e) State and prove a "co-universal" definition for $D^{-1}R$.

5. Let R be an integral domain and $P \subseteq R$ a prime ideal.

   (a) Show that both P and $R \setminus P$ are multiplicative semigroups.
   (b) Let $R_P := D^{-1}R$ be the ring of fractions for $D = R \setminus P$. Show that $U(R_P) = R_P \setminus R_P P$. Conclude that $R_P P$ is the unique maximal ideal in R_D.
   (c) Describe this construction for the prime ideal $(p) \subseteq \mathbb{Z}$, and then for $(x) \subseteq \mathbb{Z}[x]$.