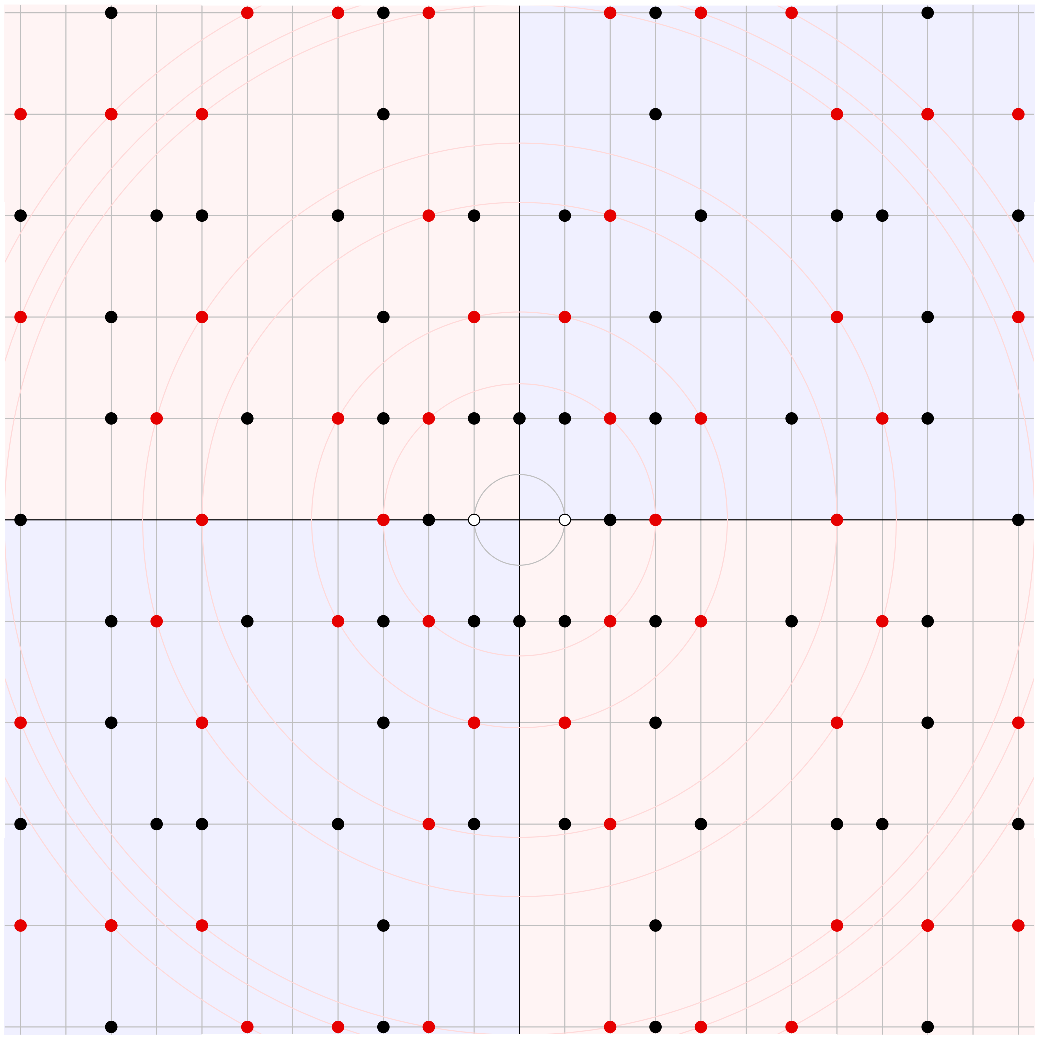
1. A picture illustrating the quadratic integers $R_{-5} = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$ as a subring of $\mathbb{C}$ is shown below, with the primes in black, and non-prime irreducibles in red.



   (a) Create an analogous picture for the ring $R_{-6}$. Start with a blank diagram with the norms of the quadratic integers labeled at each correponding lattice point.

   (b) Give an elementary characterization of non-prime irreducibles.

2. Consider the quadratic field $K = \mathbb{Q}(\sqrt{m})$ and an odd prime $p \in \mathbb{Z}$.

   (a) Show that if $p \mid m$, then $p$ is ramified in $R_m$, by establishing
$$(p) = (p, \sqrt{m})^2.$$

   (b) Show that if $p \nmid m$ and $m \equiv n^2 \pmod p$, then $p$ splits in $R_m$, via
$$(p) = (p, n + \sqrt{m})(p, n - \sqrt{m}).$$

   (c) Show that if $p \nmid m$ and $m$ is not a quadradic residue mod $p$, then $R_m/(p)$ is a field, and hence $p$ is inert in $R_m$.

3. Prove that if $m = -3$, $-7$, or $-11$, then $R_m$ is Euclidean with $d(r) = |N(r)|$ for all nonzero $r \in R_m$. [*Hint*: Mimic the proof of the same result for $m = -2$, $-1$, $2$, and $3$, but choose $d \in \mathbb{Z}$ nearest to $2t$ and then $c \in \mathbb{Z}$ so that $c$ is as near to $2s$ as possible with $c \equiv d \pmod 2$, then set $q = (c + d\sqrt{m})/2$.]

4. Suppose $P \neq 0$ is a prime ideal in the ring $R_m$ of quadratic integers.

   (a) Show that $P \cap \mathbb{Z}$ is a prime ideal in $\mathbb{Z}$, so $P \cap \mathbb{Z} = (p)$ for some prime $p$ in $\mathbb{Z}$.

   (b) Set $I = pR_m \subseteq P$ and form the quotient ring $R/I$. Show that $R/I$, as an additive group, is generated by two elements of finite order; hence $R/I$ is finite.

   (c) Show that there is an epimorphism $R/I \twoheadrightarrow R/P$ and conclude that $R/P$ is finite.

   (d) Conclude that every prime ideal in $R_m$ is maximal.

5. An element $e \in R$ is called an *idempotent* if $e^2 = e$, and two nonzero idempotents $e_1, e_2$ are called an *orthogonal pair* if $e_1 + e_2 = 1$ and $e_1 e_2 = 0$.

   (a) Show that the following are equivalent:

      (i) $R$ contains an idempotent different from 0 and 1.

      (ii) $R$ contains an orthogonal pair of idempotents.

      (iii) $R \cong R_1 \times R_2$ for some rings $R_1$ and $R_2$.

   (b) Give an example of a non-orthogonal pair of distinct idemponents.

   (c) Find all idempotents in the ring $\mathbb{Z}/20\mathbb{Z}$.