1. (a) Solve the congruences

$$x \equiv 1 \pmod{8}, \qquad x \equiv 3 \pmod{7}, \qquad x \equiv 9 \pmod{11}$$

simultaneously for x in the ring $\mathbb{Z}$ of integers.

(b) Solve the congruences

$$x \equiv i \pmod{i+1}, \qquad x \equiv 1 \pmod{2-i}, \qquad x \equiv 1+i \pmod{3+4i}$$

simultaneously for x in the ring R_{-1} of Gaussian integers.

(c) Solve the congruences

$$f(x) \equiv 1 \pmod{x-1}, \qquad f(x) \equiv x \pmod{x^2+1}, \qquad f(x) \equiv x^3 \pmod{x+1}$$

simultaneously for $f(x)$ in $F[x]$, where F is a field in which $1+1 \neq 0$.

2. Let R be a commutative ring with identity and $\mathbb{F}$ a field.

(a) Show that if R is a PID then any nonzero prime ideal $P \subseteq R$ is a maximal.

(b) Show that there is a bijective correspondence between maximal ideals of $\mathbb{F}[x]$ and monic irreducible polynomials in $\mathbb{F}[x]$.

(c) Show that if $M \subsetneq \mathbb{Z}[x]$ is a maximal ideal, then $M \bigcap \mathbb{Z} = (p)$ for some prime $p \neq 0$.

(d) Show that there is a bijective correspondence between maximal ideals of $\mathbb{Z}[x]$ that contain p and monic irreducible polynomials in $\mathbb{Z}_p[x]$.

(e) Characterize all maximal ideals of $\mathbb{Z}[x]$.

3. Consider the following rings R_i, for $i = 1, \dots, 6$, which are additionally $\mathbb{C}$-vector spaces:

$R_1 = \mathbb{C}[x]/(x^3 - 1)$

$R_2 = \mathbb{C} \times \mathbb{C} \times \mathbb{C}$

$R_3 =$ the ring of upper triangular 2×2 matrices over $\mathbb{C}$

$R_4 = \mathbb{C}[x]/(x-1) \times \mathbb{C}[x]/(x+i) \times \mathbb{C}[x]/(x-i)$

$R_5 = \mathbb{C}[x]/(x^2+1) \times \mathbb{C}[x]/(x-1)$

$R_6 = \mathbb{C}[x]/(x+1)^2 \times \mathbb{C}[x]/(x-1)$.

(a) Compute the dimension of each R_i as a $\mathbb{C}$-vector space by giving an explicit basis.

(b) Partition the rings $R_1, \dots, R_6$ into isomorphism classes.

4. For a fixed $a \in R$, denote the polynomial *evaluation map* by

$$\phi_a \colon R[x] \longrightarrow R, \qquad \phi_a \colon f(x) \longmapsto f(a).$$

(a) If $I = (3, x^2+1)$ in $\mathbb{Z}[x]$, show that $\phi_a(I) = \mathbb{Z}$.

(b) Show by example how Part (c) can fail if 3 is replaced with a different odd prime p.

(c) Given a polynomal, $f(x) \in R[x]$, substituting $a \in R$ for x determines a *polynomial function* $f \colon R \to R$, where $a \mapsto f(a)$. Show that if R is an infinite integral domain, then the mapping $f(x) \mapsto f$ assiging to each polynomial in $R[x]$ to its corresponding polynomial function is 1–1.