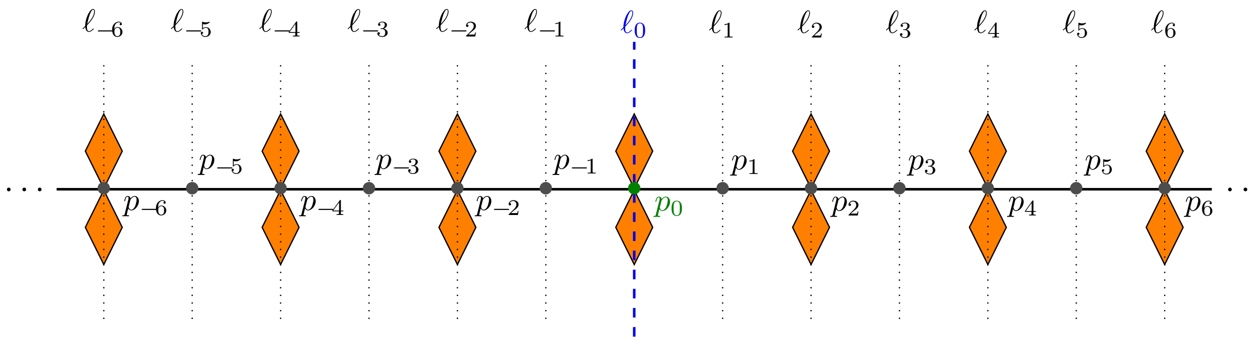
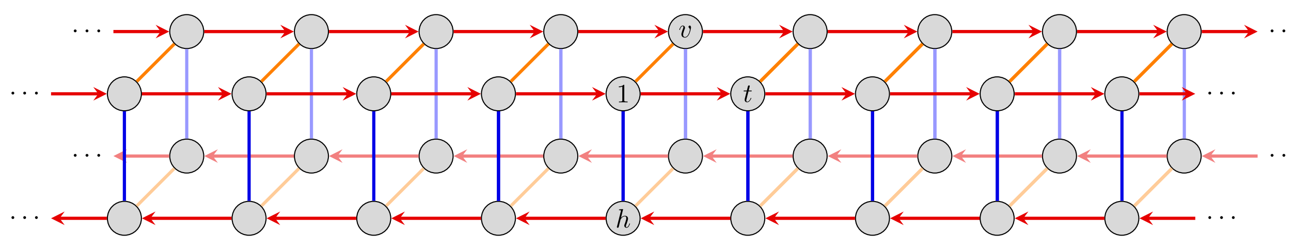
1. Consider the frieze shown below:



Let t be a minimal translation to the right, h_i a reflection across ℓ_i, and r_j a 180° rotation around p_j. Let v be the vertical reflection and $g_i = t^i v$ a glide reflection. A presentation for the frieze group is

$$\mathbf{Frz}_1 := \langle t, h, v \mid v^2 = h^2 = 1,\ th = ht^{-1},\ tv = vt,\ hv = vh \rangle,$$

where $h = h_0$. A Cayley graph is shown below.



(a) Every symmetry is either a translation t^i, glide reflection g_j, rotation r_k, horizontal reflection h_ℓ, or the vertical reflection v. Label the vertices of this Cayley graph with elements written in this form.

(b) Now, consider the generating set $\mathbf{Frz}_1 = \langle g, h, r \rangle$, where $g = g_1$ and $r = r_0$. Construct a Cayley graph, write down the presentation, and then repeat Part (a) for this Cayley graph.

(c) Characterize the *conjugacy classes* of the reflections and rotations by determining which symmetries $t^i h t^{-i}$ and $t^i r t^{-i}$ are for each $i \in \mathbb{Z}$. Then determine

$$\mathrm{cl}_G(h) := \{ xhx^{-1} \mid x \in \mathbf{Frz}_1 \}, \qquad \text{and} \qquad \mathrm{cl}_G(r) := \{ xrx^{-1} \mid x \in \mathbf{Frz}_1 \}.$$

(d) Characterize the conjugacy classes of the translations and glide reflections. That is, find

$$\mathrm{cl}_G(t^i) := \{ xt^i x^{-1} \mid x \in \mathbf{Frz}_1 \}, \qquad \text{and} \qquad \mathrm{cl}_G(g_i) := \{ xg_i x^{-1} \mid x \in \mathbf{Frz}_1 \}.$$

2. Let G be a finite group.

   (a) Show that for any $n > 2$, the number of elements in G of order n is even.

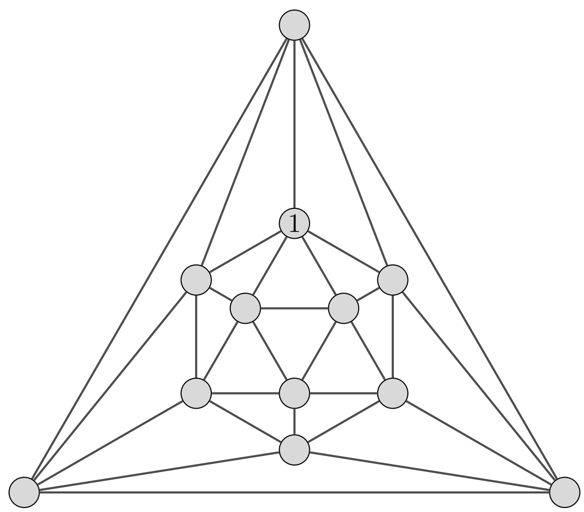
   (b) If $|G|$ is even, show that G has an element of order 2.

   (c) Give explicit examples to show how the conclusions of the previous parts can fail if the hyoptheses are not met: $n = 2$ for Part (a) and odd $|G|$ for Part (b).

   (d) Show that if $g^2 = 1$ for all $g \in G$, then G must be abelian. Does this hold for infinite groups?

3. List all fifteen abelian groups of order $432 = 2^4 \cdot 3^3$ writing each one as a product of cyclic groups of prime power order. Then, determine which group it is isomorphic to of the form $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$, where $n_{i+1} \mid n_i$; this is the "elementary divisor" form.

4. Draw the Cayley graph of the group $G = \langle a, b, c \mid a^2 = b^3 = c^3 = abc = 1 \rangle$ on the skeleton of the icosahedron, shown below, and label the nodes with elements written using a, b, and c.



   There are five groups of order 12: the abelian groups C_{12} and $C_6 \times C_2$, the dihedral group D_6, the alternating group A_4, and the dicyclic group Dic_6. Determine which group G is isomorphic to, and then re-draw this Cayley graph with the nodes labeled with elements of that group.

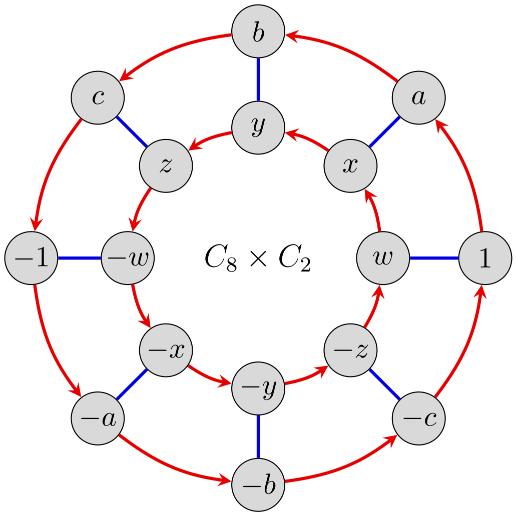
5. Cayley graphs for six groups of order 16 are shown below. Carry out the following steps for each.

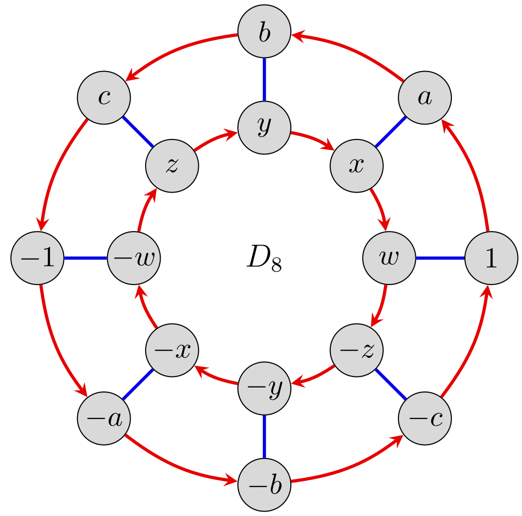
   (a) Write down a group presentation.

   (b) Identifying each element with its "negative" yields a quotient group of order 8. Construct a Cayley table and Cayley graph for these quotient, using the elements

   $$\pm 1, \ \pm a, \ \pm b, \ \pm c, \ \pm w, \ \pm x, \ \pm y, \ \pm z,$$

   and determine the resulting isomorphism type. If two groups give the same table and graph, you do not need to write this out twice.


b
c
a
y
z
x
-1
$-w$
$C_8 \times C_2$
w
1
$-x$
$-z$
$-y$
$-a$
$-c$
$-b$


b
c
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w
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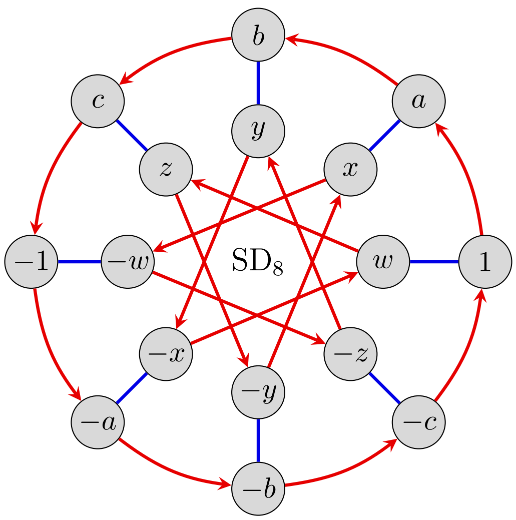
$-x$
$-z$
$-y$
$-a$
$-c$
$-b$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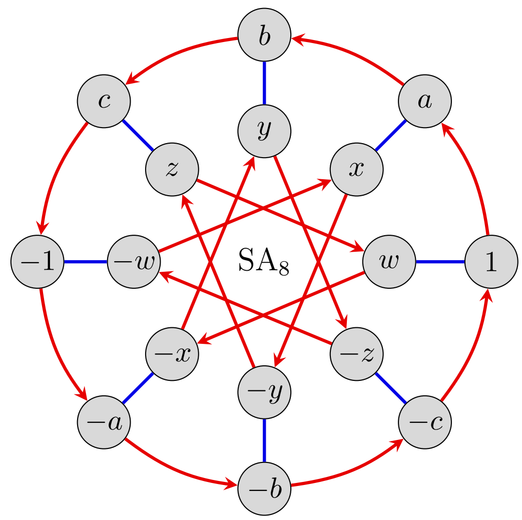
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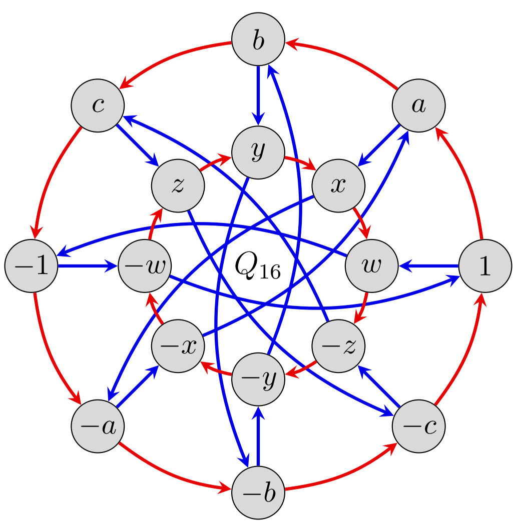
b
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Q_{16}
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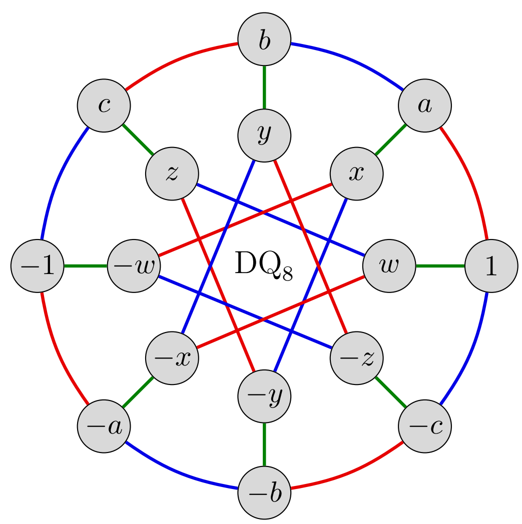
$-a$
$-c$
$-b$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DQ_8
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1
$-x$
$-z$
$-y$
$-a$
$-c$
$-b$

6. Two Cayley graphs arranged on a truncated cube are shown below. One of these is of the symmetric group S_4, and the other is $A_4 \times C_2$, which can be written as

$$A_4 \times C_2 = \{ \pm \sigma \mid \sigma \in A_4 \}.$$

Determine which Cayley graphs goes with which group, and then label the nodes with permutations in cycle notation, written as a product of disjoint cycles.