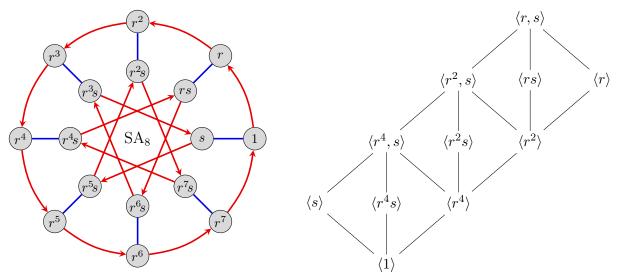
- 1. Establish the following isomorphisms by defining an explicit map and proving that it is a bijective homomorphism.
  - (a)  $H \cong xHx^{-1}$ , for any subgroup  $H \leq G$  and fixed  $x \in G$ .
  - (b)  $(\mathbb{Q}^*, \cdot) \cong (\mathbb{Q}^+, \cdot) \times C_2$ , where  $C_2 = \{1, -1\}$ .
- 2. Let G be the *semiabelian group* of order 16, defined by the presentation

$$SA_8 = \langle r, s \mid r^8 = s^2 = 1, srs = r^5 \rangle,$$

A Cayley diagram and subgroup lattice are shown below.



- (a) The subgroups  $V = \langle r^4, s \rangle$ ,  $H = \langle r^2 s \rangle$ ,  $K = \langle r^2 \rangle$ , and  $N = \langle r^4 \rangle$  are all normal. Highlight their cosets on a fresh Cayley diagram by colors.
- (b) Construct a Cayley table for the quotient of G by each of these subgroups. Then draw a Cayley diagram for each, labeling the nodes with elements (i.e., cosets).
- (c) Let  $N = \langle r^4 \rangle$ . The shaded region below shows an order-4 cyclic subgroup of G/N, generated by the element rN, and how the union of these four cosets is the order-8 subgroup  $\langle r \rangle$  of G. Construct analogous tables for the other five non-trivial proper subgroups of G/N, and then draw the subgroup lattice of G/N.

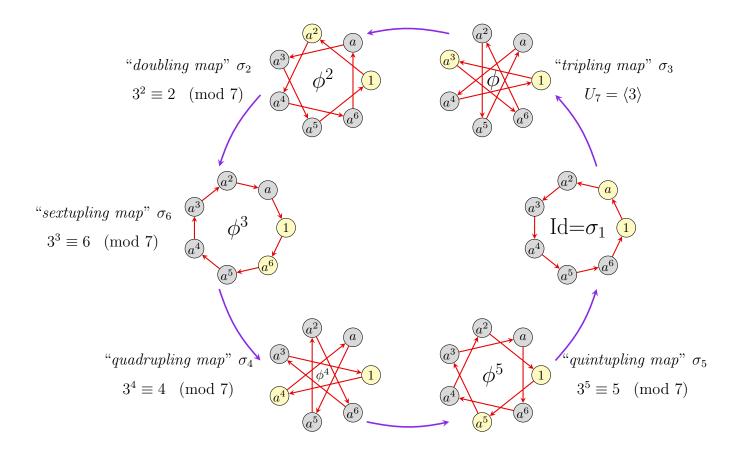
$r^3N$	$r^3 s N$		$r^3$	$r^7$	$r^3s$	$r^7s$		$r^3$	$r^7$	$r^3s$	$r^7$
$r^2N$	$r^2 s N$		$r^2$	$r^6$	$r^2s$	$r^6s$		$r^2$	$r^6$	$r^2s$	$r^6$
rN	rsN		r	$r^5$	rs	$r^5s$		r	$r^5$	rs	$r^5$
Ν	sN		1	$r^4$	s	$r^4s$		1	$r^4$	s	$r^4$
$\langle rN \rangle \leq G/N$		-	$\langle r \rangle / N \le G / N$					$\langle r \rangle \leq G$			

(d) Repeat the previous part for the subgroups H, K, and V, but include the trivial and nonproper subgroups.

- 3. Suppose  $A, B \leq G$ , and that A normalizes B. That is,  $A \leq N_G(B)$ .
  - (a) Show that  $AB \leq G$ .
  - (b) Show that  $B \trianglelefteq AB$  and  $A \cap B \trianglelefteq A$ .
  - (c) Show that  $A/(A \cap B) \cong AB/B$ .
  - (d) Show that if A and B are both normal with G = AB, then

 $G/(A \cap B) \cong (G/A) \times (G/B).$ 

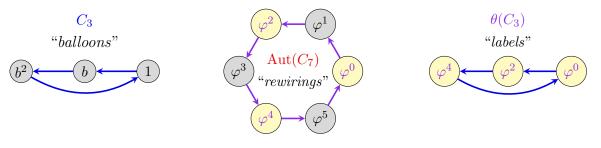
- 4. Prove the remaining parts of the correspondence theorem. That is, if  $N \leq H \leq G$  is a chain of subgroups and  $N \leq G$ , then show all of the following.
  - (a)  $H/N \leq G/N$  if and only if  $H \leq G$
  - (b) [G/N:H/N] = [G:H]
  - (c)  $H/N \cap K/N = (H \cap K)/N$
  - (d)  $\langle H/N, K/N \rangle = \langle H, K \rangle / N$
  - (e) H/N is conjugate to K/N in G/N if and only if H is conjugate to K in G.
- 5. The semidirect product  $C_7 \rtimes C_3$  can be constructed in a "visual" manner by starting with a Cayley graph of Aut $(C_7) \cong U_6 \cong C_6$ , shown below, with the nodes labeled by "rewirings" of a Cayley graph of  $C_7$ .



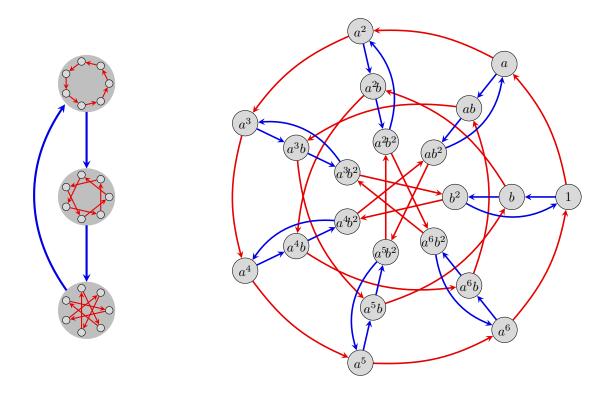
Next, we define the "labeling map," a homomorphism

$$\theta \colon C_3 \longrightarrow \operatorname{Aut}(C_7), \qquad \theta \colon b^k \longmapsto \phi^{2k},$$

which tells us how to "stick in" rewired copies of  $A = C_7$  into "inflated" nodes of  $B = C_3$ .



By connecting up the corresponding nodes, we get a Cayley graph for  $C_7 \rtimes C_3$ , like the following.



- (a) Carry out analogous steps to construct Cayley graphs of  $C_9 \rtimes C_3$  and  $C_3 \rtimes C_6$ , and then write down a presentation for each group. [*Hint*: There are two homomorphisms  $C_3 \rightarrow \operatorname{Aut}(C_9)$  that will work, but one of them leads to a much less tangled diagram.]
- (b) In each case, this group is isomorphic to the Cartesian product  $G = A \times B$  with a different binary operation than the direct product. Give an explicit formula for  $(a^i, b^j) * (a^k, b^\ell)$  using this new binary operation.