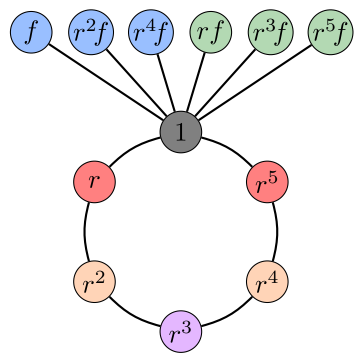
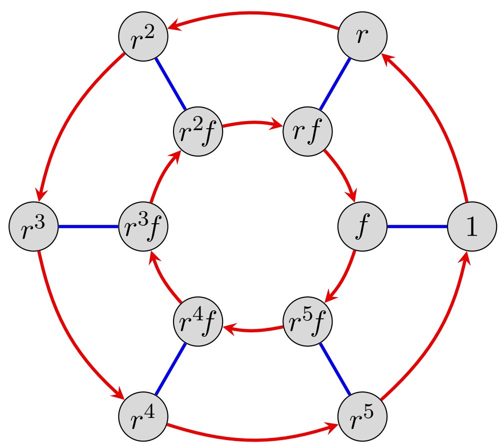
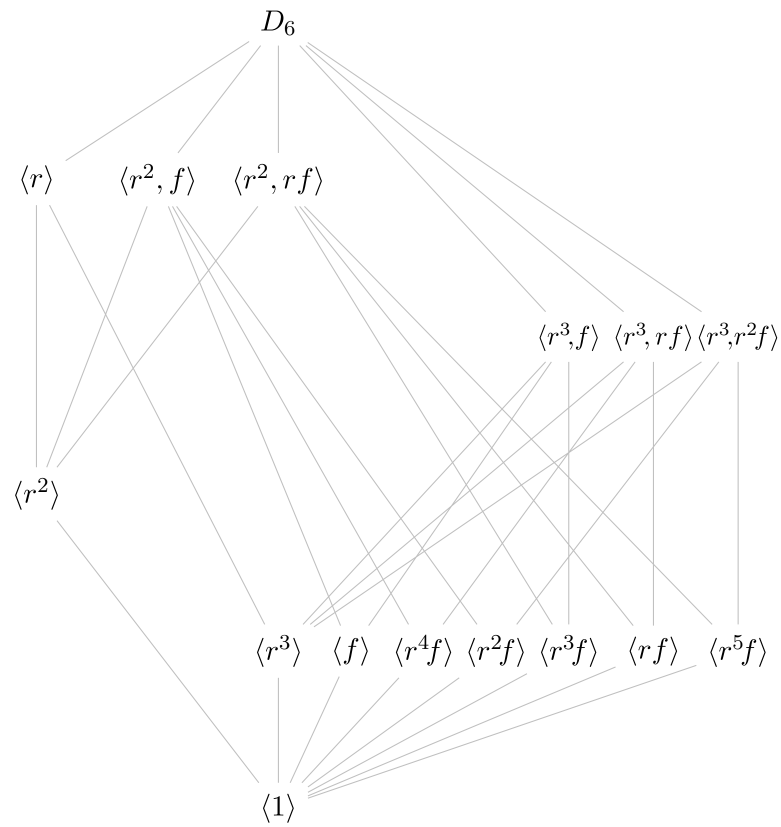
1. In this multi-part adventure, we will explore actions of the group $G = D_6 = \langle r, f\rangle$ on various sets of objects, and its automorphism group. A Cayley graph, cycle graph, and subgroup lattice are shown below.

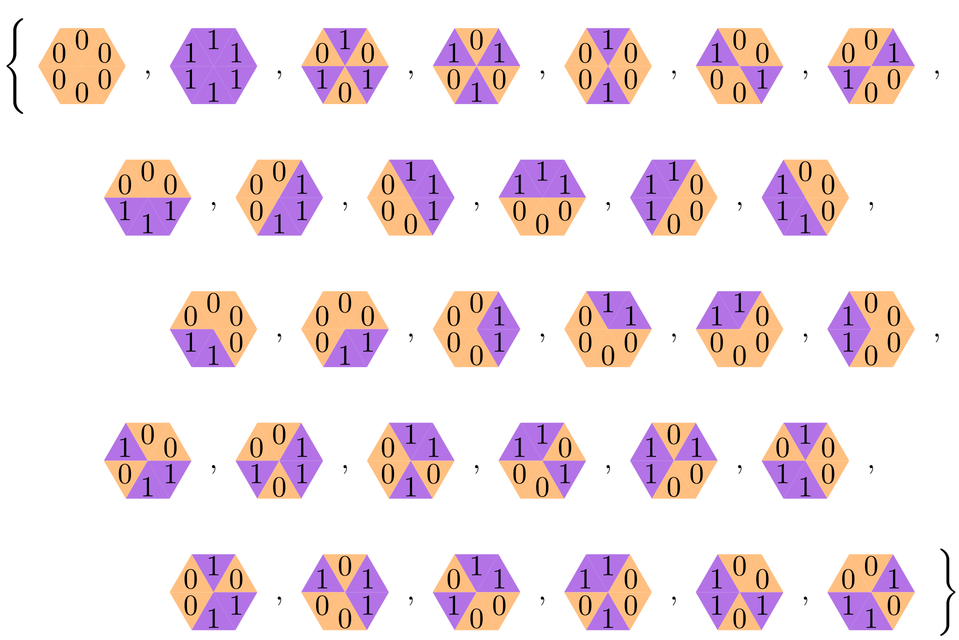






(a) Consider the right action of D_6 on following set of 31 "binary hexagons," where r rotates each one $60°$ counterclockwise, and f flips it across a vertical axis.

$S =$



(i) Draw the action graph and construct the "*fixed point table*", which has a checkmark in row g and column s if $\phi(g).s = s$.

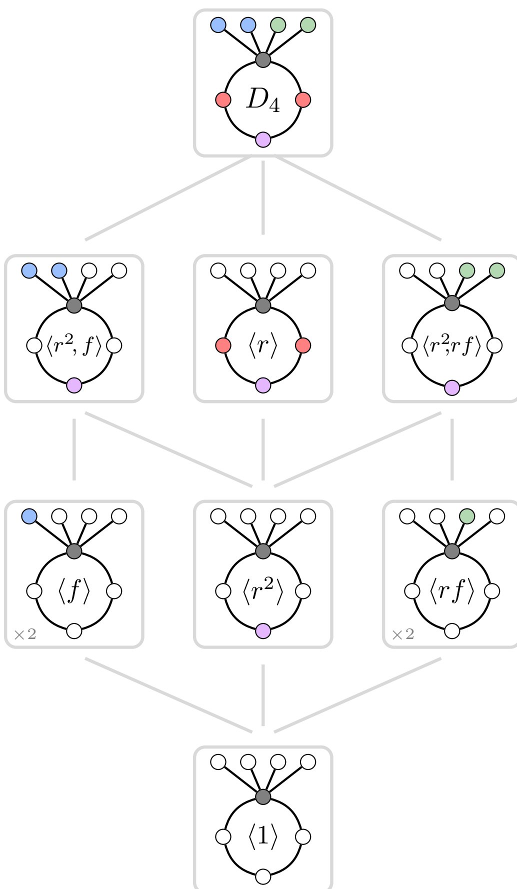
(ii) Next to each $s \in S$ on your action graph, write $\text{stab}(s)$, the stabilizer subgroup, using its generators.

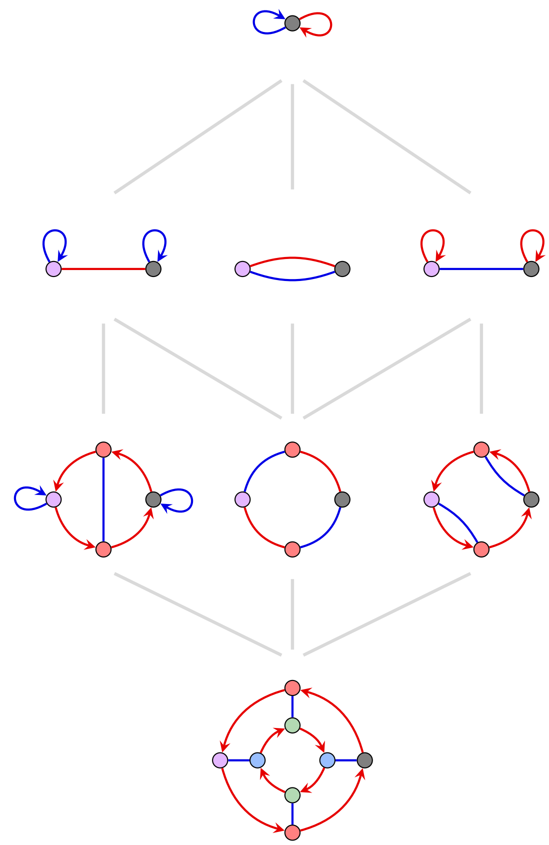
(iii) Find $\text{fix}(g)$ for each $g \in D_6$.

(iv) Find $\text{Ker}(\phi)$ and $\text{Fix}(\phi)$.

(b) Let $G = D_6$ act on its subgroups by conjugation. Construct the action graph by superimposing it on the subgroup lattice, and then construct the fixed point table. Label the nodes by stabilizer, and find $\text{Ker}(\phi)$ and $\text{Fix}(phi)$.

(c) There is a *Galois correspondence* between conjugacy classes of subgroups of G and transitive actions of G, defined by collapsing the right cosets of H. An example of this correspondence for $D_4 = \langle r, f\rangle$ is shown below. Carry out this construction for $D_6 = \langle r, f\rangle$.

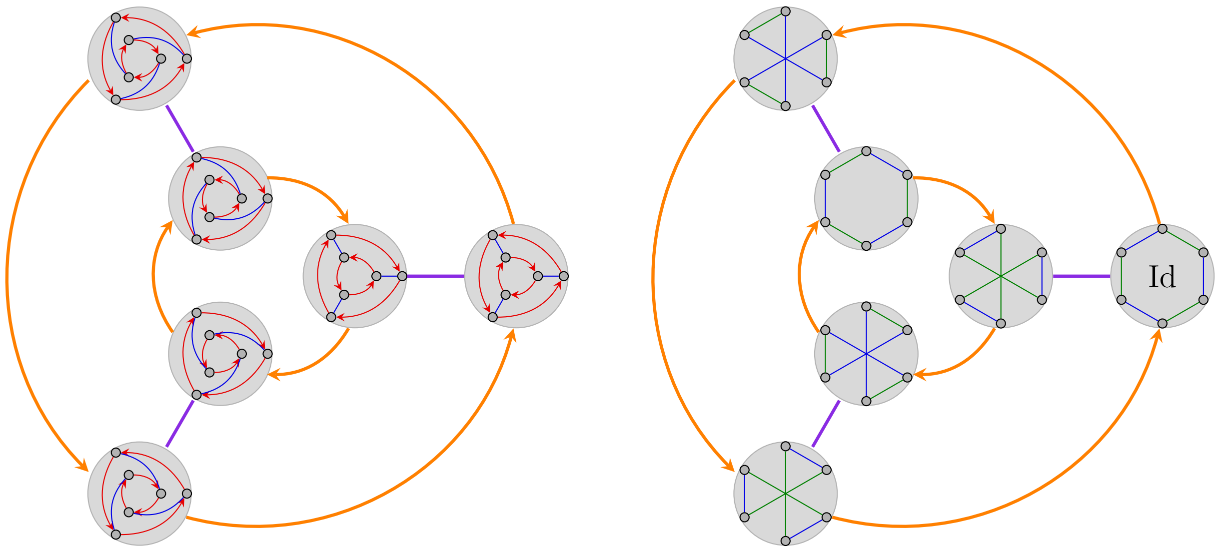




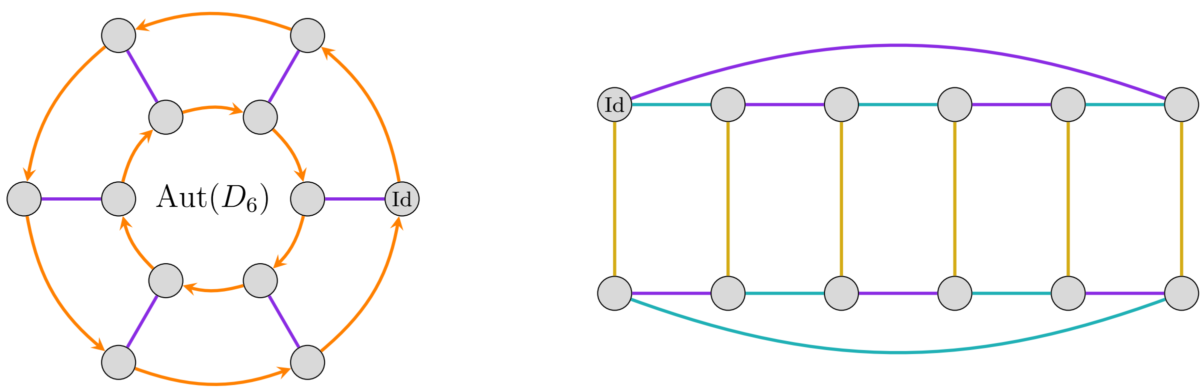
(d) The automorphism group of D_3 is $\text{Aut}(D_3) = \langle \alpha, \beta\rangle \cong D_3$, where

$$\begin{cases} \alpha(r) = r \\ \alpha(f) = rf \end{cases} \qquad \begin{cases} \beta(r) = r^{-1} \\ \beta(f) = f \end{cases}$$

Below are two Cayley graphs for the automorphism group $\text{Aut}(D_3) \cong D_3$ with the nodes labeled by rewired Cayley graphs of $D_3 = \langle r, f\rangle$ and $D_3 = \langle s, t\rangle = \langle f, rf\rangle$.



Construct two Cayley graphs for $\mathrm{Aut}(D_6)$ labeled with rewirings, using one or both of the following Cayley graphs. (That is, there are four possibilities, pick two of them to do.)



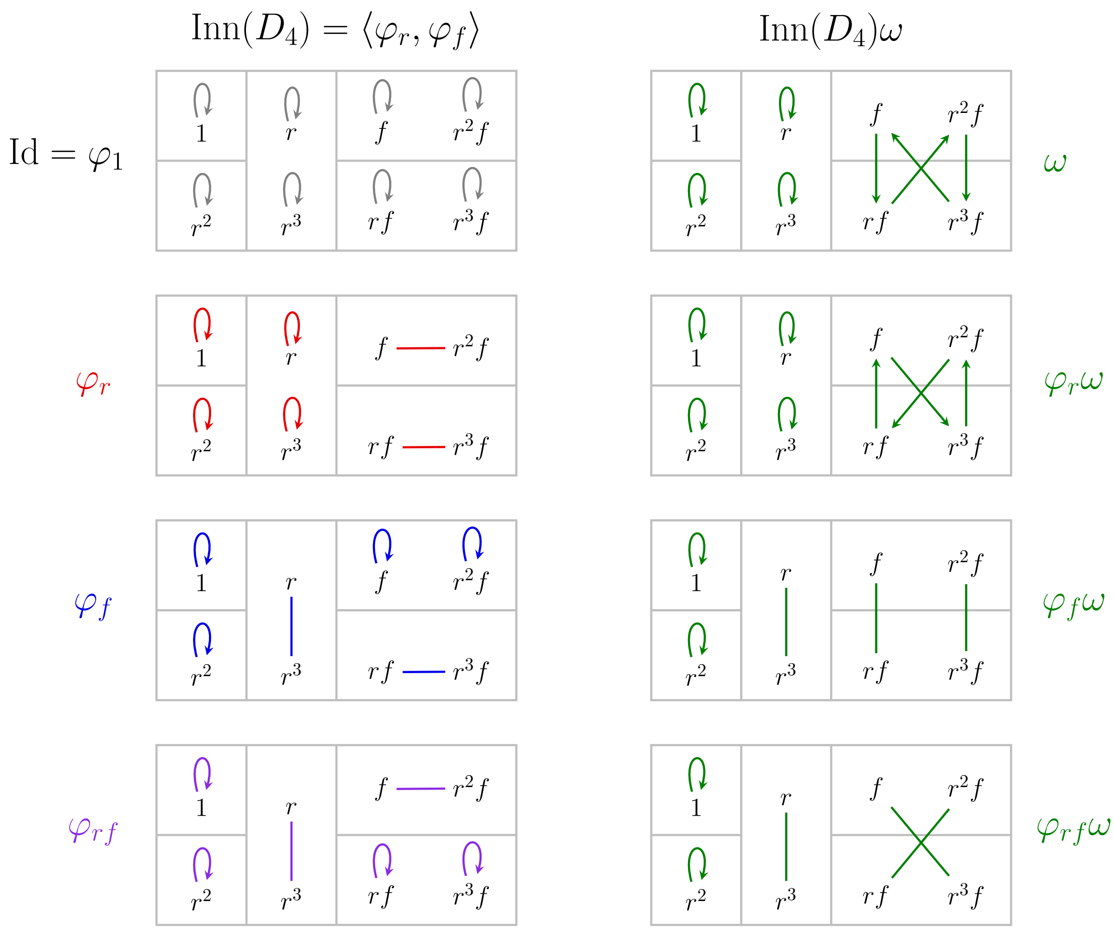
(e) Half of the automorphisms of D_6 are *inner*, which means they have the form $\varphi_x\colon g \mapsto x^{-1}gx$ for some $x \in D_6$. Let $\omega \in \mathrm{Aut}(D_6)$ be the outer automorphism

$$\omega\colon D_6 \longrightarrow D_6, \qquad \omega(r) = r, \quad \omega(f) = rf$$

of order 6 that cyclically rotates axes of reflections of the square. The automorphism group is the union of cosets

$$\mathrm{Aut}(D_6) = \mathrm{Inn}(D_6) \cup \mathrm{Inn}(D_6)\omega \cong \mathrm{Inn}(D_6) \rtimes \mathrm{Out}(D_6).$$

This partition, but for D_4, is shown below. Construct an analogous diagram for D_6.



(f) Find an outer automorphism $\eta \in \mathrm{Aut}(D_6)$ such that $\mathrm{Aut}(D_6) \cong \mathrm{Inn}(D_6) \rtimes \langle \eta \rangle$, and then construct the subgroup lattice of $\mathrm{Aut}(D_6)$.

(g) Go back to the Cayley graphs for $\mathrm{Aut}(D_6)$ constructed in Part (d), and label the large nodes with the corresponding automorphisms.

(h) Construct the action graph and fixed point table of the action of $\mathrm{Aut}(D_6)$ on D_6. Find $\mathrm{Ker}(\phi)$ and $\mathrm{Fix}(\phi)$.

(i) Construct the action graph and fixed point table of the action of $\mathrm{Aut}(D_6)$ on the conjugacy classes of D_6. Find $\mathrm{Ker}(\phi)$ and $\mathrm{Fix}(\phi)$.

(j) Revisit the action graphs of D_6 that you constructed via the Galois correspondence. Have any of them not appeared in this problem thus far as an orbit? If so, define an action of D_6 that has it as its action graph.

2. Let $N, K \leq G$. Show that if the following conditions hold:

(a) N is normal in G (b) $K \cap N = \{e\}$ (c) $G = KN$.

then $G \cong K \ltimes_\theta N$. Define the specific homomorphism $\theta \colon K \to \mathrm{Aut}(N)$ and show why this works.

3. Show that $\mathrm{Inn}(G)$ is a subgroup of $\mathrm{Aut}(G)$, that it is normal, and that it is isomorphic to $G/Z(G)$.

4. Suppose a group G of order 55 acts on a set S of size 14. Let $s \in S$ be an arbitrary element.

   (a) What are the possible sizes of the orbit of s?

   (b) What are the possible sizes of the stabilizer of s?

   (c) Show that this action must have a fixed point.

   (d) What is the fewest number of fixed points that this action can have? Justify your answer.

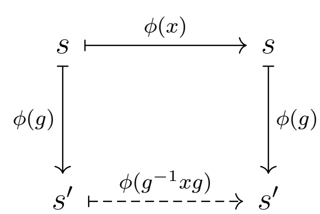
5. Let $\phi\colon G \to \mathrm{Perm}(S)$ be a group action.

   (a) Prove the orbit-stabilizer theorem for left actions by constructing a bijection between $\mathrm{orb}(s)$ and left cosets of $\mathrm{stab}(s)$. Use the same notational conventions from lecture.

   (b) We saw that elements in the same orbit have conjugate stabilizers. Explicitly, for right actions, this is

$$\mathrm{stab}(s.\phi(g)) = g^{-1}\,\mathrm{stab}(s)g, \quad \text{for all } g \in G \text{ and } s \in S,$$

   and is summarized by the following commutative diagram:



   Formulate and prove an analogous statement for left actions.