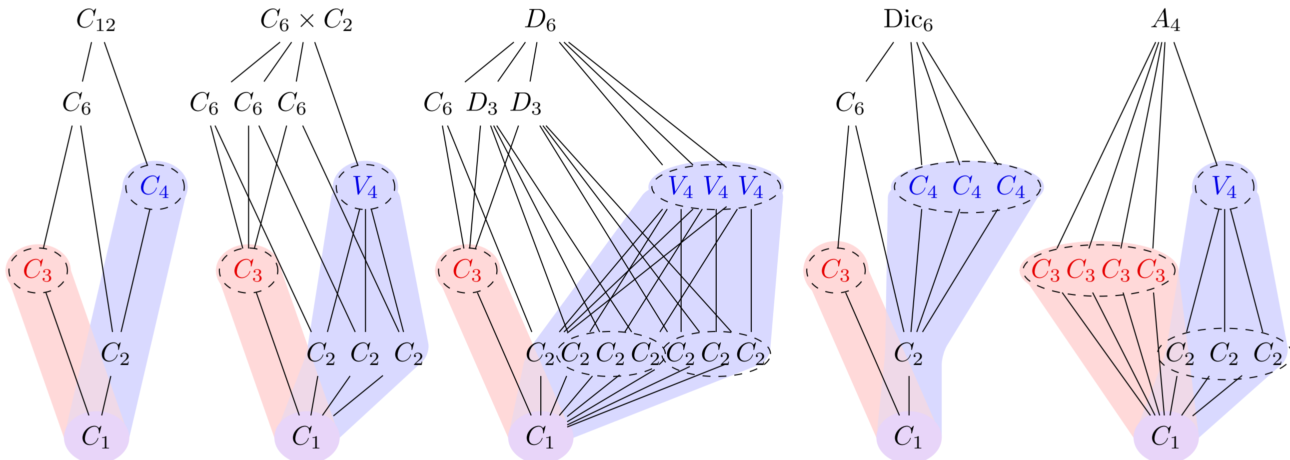
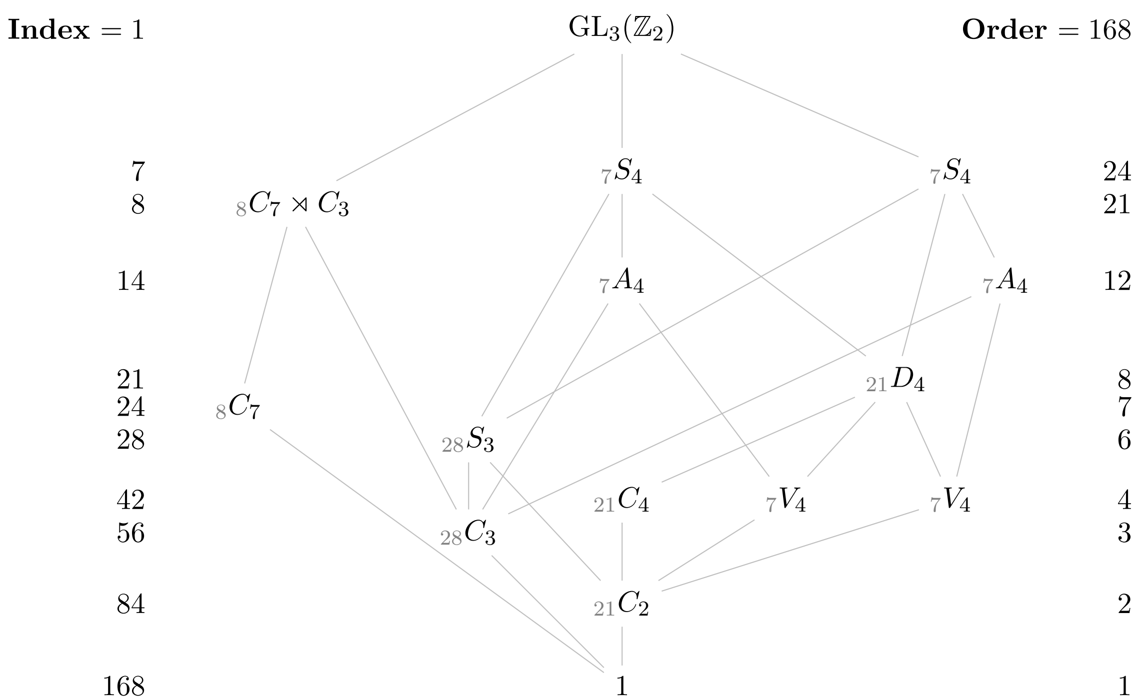
1. Loosely speaking, the Sylow theorems tell us that (1) all p-subgroups come in a single "p-subgroup tower", (2) the "top" of these towers are a single conjugacy class, and (3) the size of this class is 1 mod p. This is illustrated below with the groups of order 12.



Using the LMFDB, construct analogous diagrams for the groups of order 18 and 20.

2. After A_5, the next smallest nonabelian simple group is $G = \mathrm{GL}_3(\mathbb{Z}_2)$, the invertible 3×3 binary matrices. It has order $168 = 2^3 \cdot 3 \cdot 7$, and its reduced subgroup lattice is below.



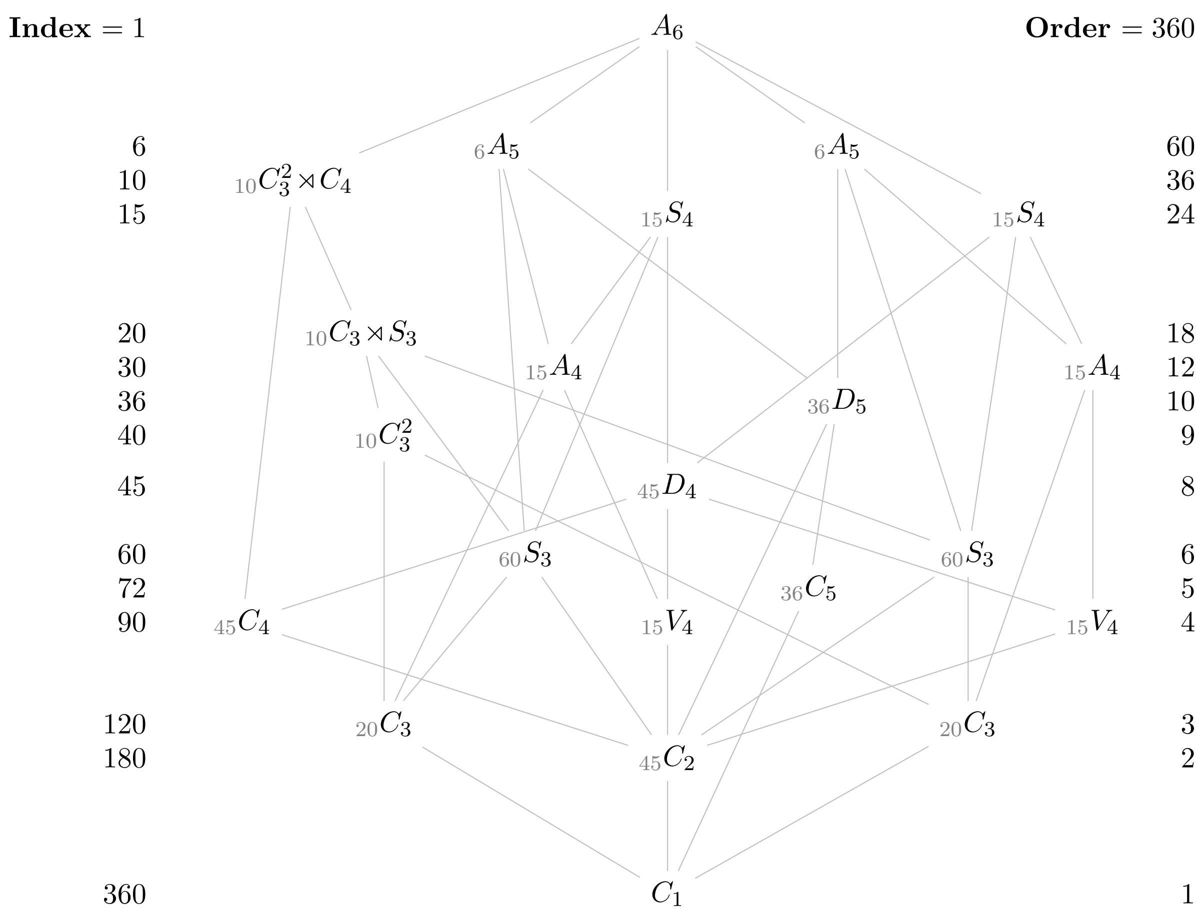
(a) Color-code the p-subgroups, then draw arrows from each $\mathrm{cl}(H)$ to $\mathrm{cl}(N(H))$.

(b) Show that there is a non-trivial homomorphism $\phi\colon \mathrm{GL}_3(\mathbb{Z}_2) \to S_8$.

(c) Show that this homomorphism must be an embedding, and conclude that the order-40320 group S_8 has at least one subgroup isomorphic to $\mathrm{GL}_3(\mathbb{Z}_2)$.

(d) Show that every such subgroup of S_8 additionally must be contained in A_8.

3. The alternating group A_6 is the third smallest nonabelian simple group. It has order $6!/2 = 360 = 2^3 \cdot 3^2 \cdot 5$, and 501 subgroups contained in 22 conjugacy classes.



(a) Distinguish the p-subgroups by colors on the lattice.

(b) For each non-singleton conjugacy class $\mathrm{cl}(H)$, draw an arrow from it to $\mathrm{cl}(N(H))$, the conjugacy class of its normalizer.

(c) Now, let G be an unknown group of order $90 = 2 \cdot 3^2 \cdot 5$.

  (i) Show that if G has a non-normal Sylow 5-subgroup, then there is be a non-trivial homomorphism $\phi\colon G \to S_6$.

  (ii) Show that if $\phi(G)$ is contained in the simple group A_6, then ϕ cannot be injective.

  (iii) Explain why this implies that G cannot be simple.

  (iv) Give an alternate proof that groups of order 90 are not simple, using the Sylow theorems.

  (v) Find all possibilities for n_2, n_3, and n_5, where n_p is the number of Sylow p-subgroups of G. Then, using GroupNames or LMFDB, make a list of all groups of order 90, and write down the actual vaules of n_2, n_3, and n_5 for each, as well as the isomorphism type of the Sylow 3-subgroup(s) – either C_9 or C_3^2. Does anything surprise you about this?

4. Show that there are no simple groups of the following order:

   (i) p^n, $(n > 1)$, (ii) pq, $(p, q \text{ prime})$ (iii) 56, (iv) 108.

5. Let P be a Sylow p-subgroup of G.

   (a) If $P \trianglelefteq H \trianglelefteq G$ for $H \neq P$, show that $G = NH$, where $N = N_G(P)$.

   (b) Show that if $x, y \in C_G(P)$ are conjugate in G, then they are conjugate in $N_G(P)$.

6. Let G be a group, not necessarily finite, and let A and B be subgroups of finite index, but not necessarily normal. In particular, we cannot assume that AB is a group, but as an (A, B)-double coset, it is a disjoint union of cosets of A.

   (a) Let B act on $S = A\backslash AB = \big\{Ax \mid x \in AB\big\}$ via the homomorphism

   $$\phi\colon G \longrightarrow \mathrm{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } Ax \mapsto Axg.$$

   Use the orbit-stabilizer theorem to show that $[A : A \cap B] = [AB : B]$.

   (b) Show that $[G : A \cap B] \leq [G : A][G : B]$. Give an explicit example of where the inequality is strict.

   (c) Show that there is some $N \trianglelefteq G$ contained in both A and B with $[G : N] \leq \infty$.

   (d) Use Part (a) and the Sylow theorems to show that there are no simple groups of order $96 = 2^5 \cdot 3$. [*Hint*: First, show that the intersection of two Sylow 2-subgroups must have order 16, and then consider its normalizer.]