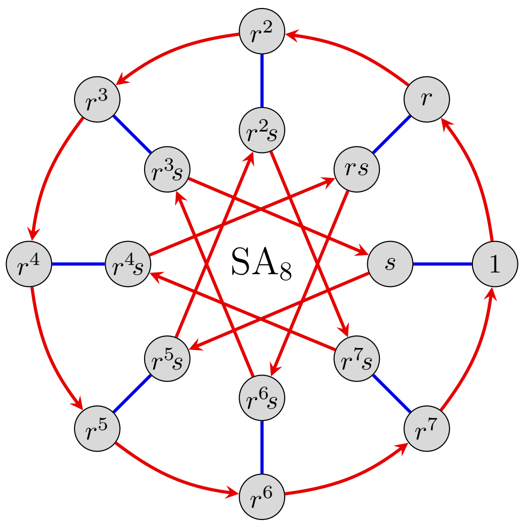
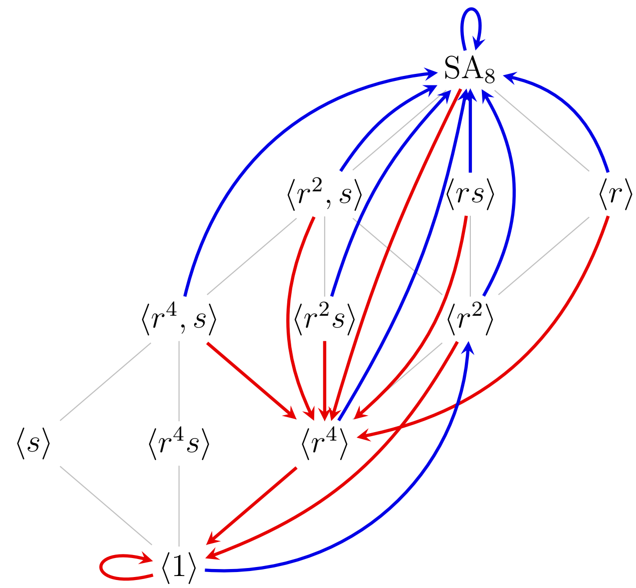
1. The *chutes and ladders diagram* of a finite group G is constructed by taking its subgroup lattice, and adding:

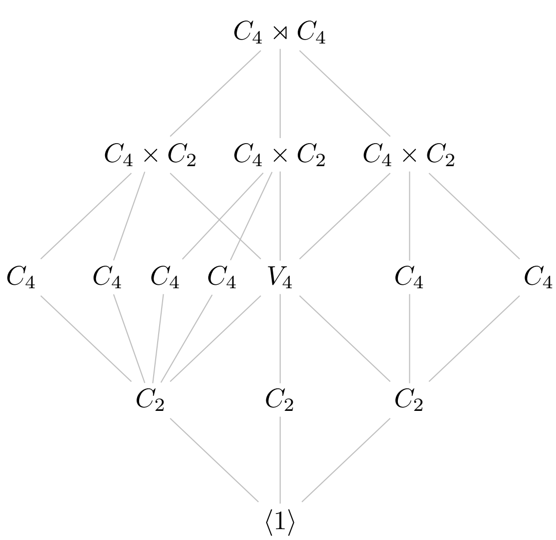
   - a red arrow for each "maximal central descent" $N \searrow L$, where $L = [G, N]$,
   - a blue arrow for each "maximal central ascent", $N \nearrow Z$, where $Z/N = Z(G/N)$.

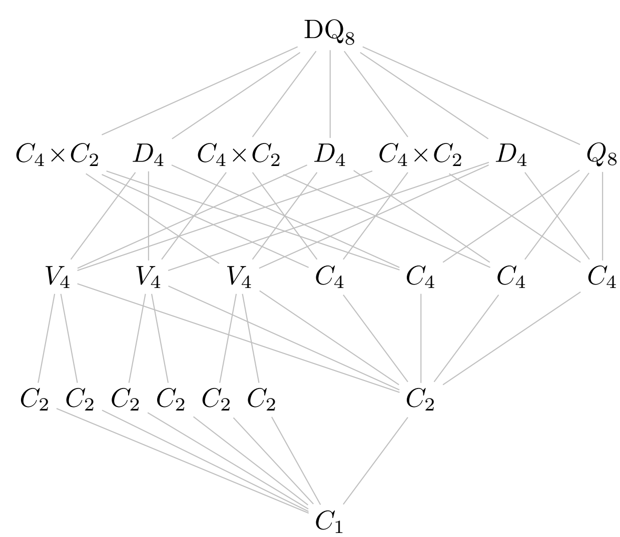
   An example is shown below for the semiabelian group $\mathrm{SA}_8 = \langle r, s \mid r^8 = s^2 = 1,\ srs = r^5 \rangle$.





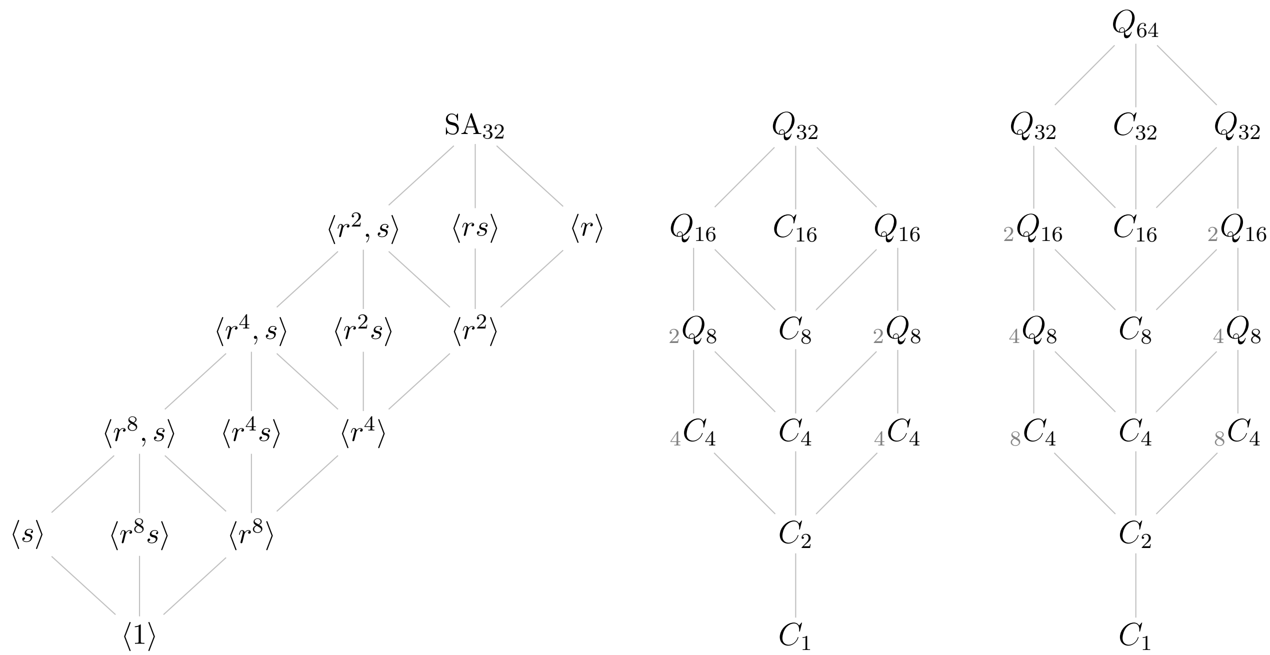
   (a) Construct the chutes and ladders diagram of the following groups: $C_4 \rtimes C_4$, DQ_8, SA_{32}, Q_{32}, and Q_{64}.





   (b) On the subgroups lattices of the groups from Part (a), mark the upper and lower central series, the derived series, and determine the abelianization.

   (c) Two functions $f : X \to Y$ and $g : Y \to X$ are *generalized inverses* if $f \circ g \circ f = f$ and $g \circ f \circ g = g$. Show that maximal central ascents and maximal central descents are generalized inverses on the set of normal subgroups of a groups G.



2. Let G be a group with nilpotency class $n < \infty$.

   (a) Show that every subgroup of G has nilpotency class at most n.

   (b) Show that every homomorphic image of G has nilpotency class at most n.

   (c) Show that nontrivial normal subgroups of G intersect the center nontrivially.

3. Let G be a finite group whose inner automorphism group $\mathrm{Inn}(G)$ is abelian. Show that $G' \leq Z(G)$, and use this to conclude that G is nilpotent.

4. Let G be a finite group in which every maximal subgroup is normal.

   (a) Prove that G is nilpotent. [*Hint*: If not, then take a non-normal Sylow subgroup $P \leq G$, and choose a maximal $M \leq G$ containing $N_G(P)$.]

   (b) Show that every maximal subgroup of G has prime index.

5. Show that if a finite group G has no fully unnormal subgroups, then it is nilpotent. [*Hint*: You may use the lemma that if P is a Sylow p-subgroup of $H \trianglelefteq G$, then $G = N_G(P)H$.]