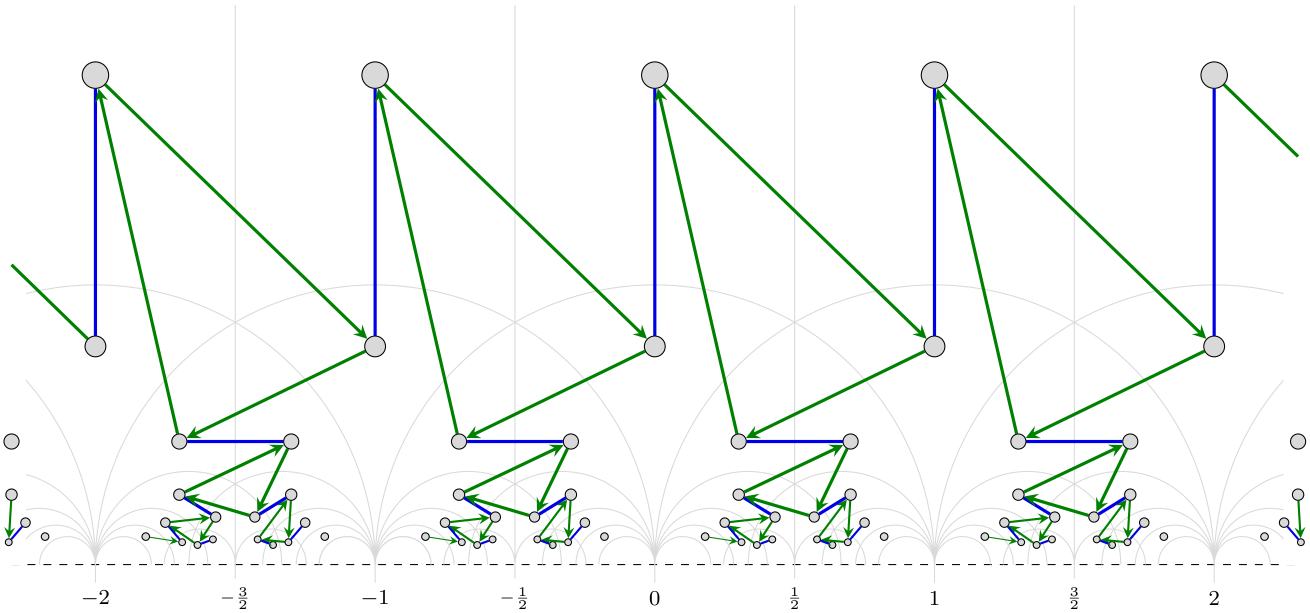
1. A Cayley graph of the *projective linear group* $\mathrm{PSL}_2(\mathbb{Z}) = \langle A, B\rangle$ is shown below.



The generators are the images $A = \pi(ST)$ and $B = \pi(S)$ under the natural quotient map $\pi\colon \mathrm{SL}_2(\mathbb{Z}) \twoheadrightarrow \mathrm{PSL}_2(\mathbb{Z})$, where

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad ST = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

Thus, we can think of them as $A = \pm ST$ and $B = \pm S$. In this problem, we will show that the identity element in $\mathrm{PSL}_2(\mathbb{Z})$ cannot be written nontrivially as

$$I = A^{i_1} B A^{i_2} B \cdots A^{i_{m-1}} B A^{i_m}, \qquad i_1, i_m \in \{0,1,2\}, \quad i_k \in \{1,2\}.$$

This will confirm that $\mathrm{PSL}_2(\mathbb{Z}) \cong C_3 * C_2$, which is suggested by the Cayley graph above.

(a) Show by slight brute force that this is impossible for $m = 1$ and $m = 2$.

(b) Now, suppose there is such a representation of the idenitity, for some $m \geq 3$. Assuming that m is minimal, left-multiply by A^{-i_1} and right-multiply by A^{i_1}. Show that $i_m + i_1$ is not a multiple of 3, and conclude that the identity element can be written as a product of BA's and BA^2's.

(c) Recalling that $A = \pm ST$ and $B = \pm S$, let $R = ST$, and consider the following matrices in $\mathrm{SL}_2(\mathbb{Z})$:

$$SR = S^2 T = -T = -\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad SR^2 = -TST = -\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

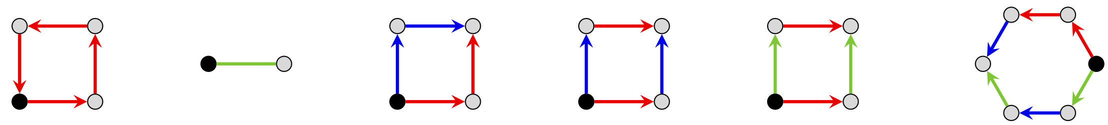
Show that for any (nontrivial) product of these matrices, the absolute sum of the entries is at least 3.

(d) Conclude that the identity element in $\mathrm{PSL}_2(\mathbb{Z})$ cannot be written nontrivially.

2. Consider the group $M = \langle S \mid R \rangle$ defined by the following presentation.

$$M = \big\langle a, b, c \mid a^4,\ c^2,\ a^2b^{-2},\ aba^{-1}b^{-1},\ aca^{-1}c^{-1},\ a^2bc^{-1}b^{-1}c^{-1} \big\rangle.$$

The relators of this presentations describe the following motifs that a Cayley graph for $M = \langle a, b, c \rangle$ must have.

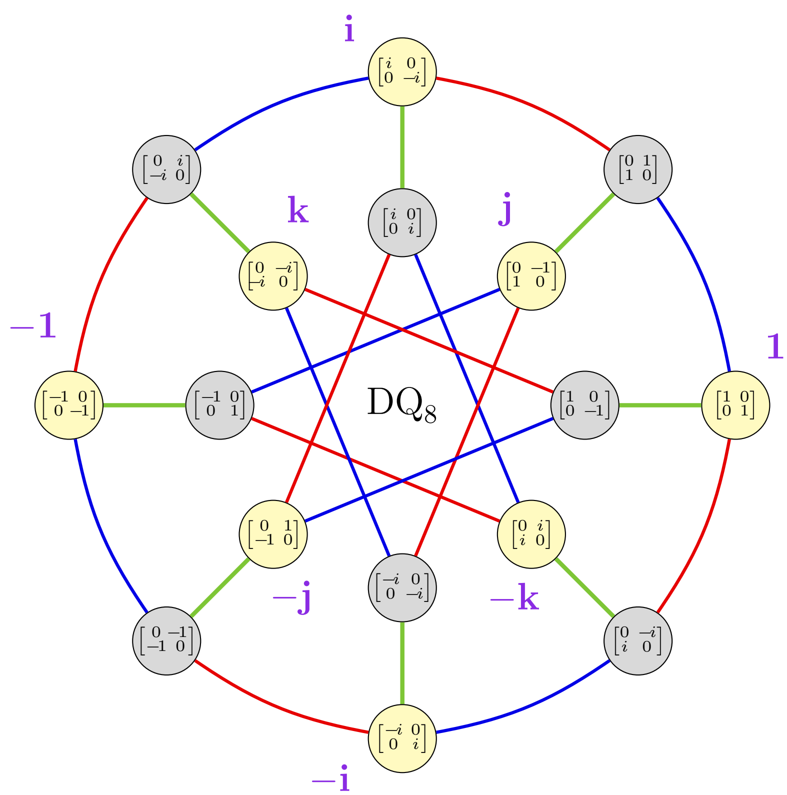


$a^4 = 1$ $\qquad$ $c^2 = 1$ $\qquad$ $a^2 = b^2$ $\qquad$ $ab = ba$ $\qquad$ $ac = ca$ $\qquad$ $a^2b = cbc$

In this problem, you will prove that this is the *diquaterion group*

$$\mathrm{DQ}_8 = \langle X, Y, Z \rangle = \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle.$$

generated by the *Pauli matrices* from quantum mechanics and information theory. A Cayley graph is shown below.



(a) Establish $|M| \leq 16$ by showing that every word in M can be written

$$a^i b^j c^k, \qquad i \in \{0, 1, 2, 3\}, \quad j \in \{0, 1\}, \quad k \in \{0, 1\}.$$

(b) Define a "relabing map" θ from S onto a generating set of matrices for DQ_8, such that $\theta(r) = I$, for every $r \in R$.

(c) Prove that $M \cong \mathrm{DQ}_8$.

(d) Construct a Cayley graph for DQ_8 with the presentation $\langle S \mid R \rangle$.

3. Prove what group is described by each presentation.

   (a) $G = \langle a, b \mid a^2 = 1,\ b^3 = 1,\ ab = ba \rangle$

   (b) $G = \langle a, b \mid a^4 = 1,\ a^2 = b^2,\ ab = ba^3 \rangle$

   (c) $G = \langle a, b \mid a^4 = b^3 = 1,\ ab = ba^3 \rangle$

   (d) $G = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (ac)^2 = (bc)^3 = 1 \rangle$.

4. If A_1, A_2, and B are objects in a category $\mathcal{C}$ with morphisms $\alpha_i \in \mathrm{Hom}_{\mathcal{C}}(A_i, B)$, then their *fiber product* (or *pullback*) is an object P with morphisms $\pi_i \in \mathrm{Hom}_{\mathcal{C}}(P, A_i)$ for which $\alpha_1 \circ \pi_1 = \alpha_2 \circ \pi_2$, such that the following property holds:

   "*For any object Q in $\mathcal{C}$ and morphisms $h_i \in \mathrm{Hom}_{\mathcal{C}}(Q, A_i)$ such that if $\alpha_1 \circ h_1 = \alpha_2 \circ h_2$, there exists a unique morphism $h \in \mathrm{Hom}_{\mathcal{C}}(Q, P)$ such that $h_i = \pi_i \circ h$ for $i = 1, 2$.*"

$$\begin{tikzcd}
Q \arrow[drr, bend left, "h_1"] \arrow[ddr, bend right, "h_2"'] \arrow[dr, dashed, "h"] & & \\
& P \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & A_1 \arrow[d, "\alpha_1"] \\
& A_2 \arrow[r, "\alpha_2"] & B
\end{tikzcd}$$

   Prove that any two pullbacks are equivalent.

5. Let A, B_1, B_2 be objects in a category $\mathcal{C}$ and let $\alpha_i \in \mathrm{Hom}_{\mathcal{C}}(A, B_i)$ for $i = 1, 2$. A *fiber coproduct* (or *pushout*) is an object C with morphisms $\iota_i \in \mathrm{Hom}(B_i, C)$ satisfying the following couniversal property:

   For any object $D \in \mathrm{Ob}(\mathcal{C})$ and morphisms $h_i \in \mathrm{Hom}_{\mathcal{C}}(B_i, D)$ such that if $h_1 \circ \alpha_1 = h_2 \circ \alpha_2$, there exists a unique $h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ such that $h \circ \iota_i = h_i$.

$$\begin{tikzcd}
A \arrow[r, "\alpha_1"] \arrow[d, "\alpha_2"'] & B_1 \arrow[d, "\iota_1"] \arrow[ddr, bend left, "h_1"] & \\
B_2 \arrow[r, "\iota_2"] \arrow[drr, bend right, "h_2"'] & C \arrow[dr, dashed, "h"] & \\
& & D
\end{tikzcd}$$

   Let Y and Z be sets with inclusion maps $\alpha_Y \colon Y \cap Z \hookrightarrow Y$ and $\alpha_Z \colon Y \cap Z \hookrightarrow Z$. Show that the pushout (or fiber coproduct) of α_Y and α_Z is equivalent to the union $Y \cup Z$, as illustrated by the following commutative diagram.

$$\begin{tikzcd}
Y \cap Z \arrow[r, hook, "\alpha_Z"] \arrow[d, hook, "\alpha_Y"'] & Z \arrow[d, hook, "\iota_Z"] \\
Y \arrow[r, hook, "\iota_Y"] & Y \cup Z
\end{tikzcd}$$