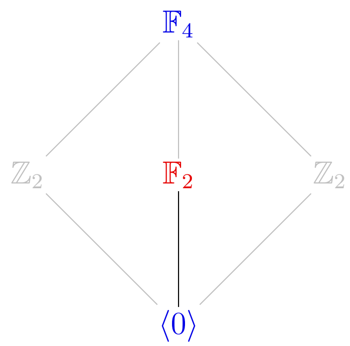
1. Construct the Cayley tables, Cayley graphs, and subring lattices of the finite field $\mathbb{F}_9 \cong \mathbb{Z}_3[x]/(x^2+x+2)$. Examples for the finite fields

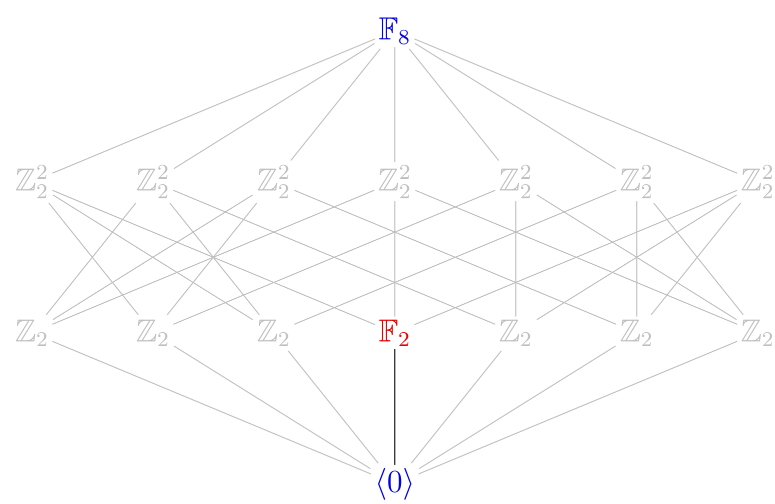
$$\mathbb{F}_4 \cong \mathbb{Z}_2[x]/(x^2+x+1) \qquad \text{and} \qquad \mathbb{F}_8 \cong \mathbb{Z}_2[x]/(x^3+x+1)$$

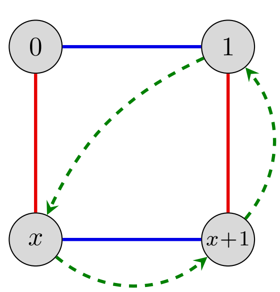
are shown below.

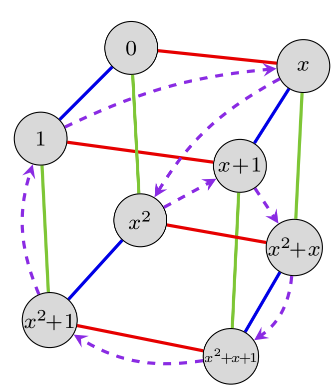
$\times$	1	x	$x+1$
1	1	x	$x+1$
x	x	$x+1$	1
$x+1$	$x+1$	1	x

$\times$	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
1	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
x	x	x^2	x^2+x	$x+1$	1	x^2+x+1	x^2+1
$x+1$	$x+1$	x^2+x	x^2+1	x^2+x+1	x^2	1	x
x^2	x^2	$x+1$	x^2+x+1	x^2+x	x	x^2+1	1
x^2+1	x^2+1	1	x^2	x	x^2+x+1	$x+1$	x^2+x
x^2+x	x^2+x	x^2+x+1	1	x^2+1	$x+1$	x	x^2
x^2+x+1	x^2+x+1	x^2+1	x	1	x^2+x	x^2	$x+1$









2. Let $f : R \to S$ be a ring homomorphism between commutative rings.
   (a) If f is surjective and I is an ideal of R, show that $f(I)$ is an ideal of S.
   (b) Show that Part (a) is not true in general when f is not surjective.
   (c) Show that if f is surjective and R is a field, then S is a field as well.

3. Use Zorn's lemma to show that the ring $\mathbb{R}$ contains a subring A containing 1 that is maximal with respect to the property that $1/2 \notin A$.

4. Let R be a commutative ring with unity.
   (a) Show that if x is contained in every maximal ideal, then $1 + x$ is a unit.
   (b) A ring is *local* if it has a unique maximal ideal. Show that R is local if and only if the non-units form an ideal.
   (c) Characterize units and maximal ideals of the following ring, where p is a fixed prime.

$$R = \left\{ \tfrac{a}{b} \mid a, b \in \mathbb{Z},\ (a, b) = 1,\ p \nmid b \right\} \subseteq \mathbb{Q},$$

5. Let R be a commutative ring with 1. The *radical* of an ideal $I \subseteq R$ is the set

$$\sqrt{I} = \left\{ x \in R \mid x^n \in I \text{ for some } n \in \mathbb{N} \right\},$$

and I is a *radical ideal* if $I = \sqrt{I}$. An ideal $I \subsetneq R$ is *primary* if $ab \in I$ implies $a \in I$ or $b^n \in I$ for some $n \in \mathbb{N}$.
   (a) Show that an ideal I is prime if and only if it is primary and radical.
   (b) Show that the radical of a primary ideal must be prime.
   (c) Show that I is primary if and only if all zero-divisors in R/I are nilpotent.
   (d) Give an example to show that, in general, there are primary ideals which are not prime powers.