1. Let R be a commutative ring with 1. In the first two problems, we will define several different "radicals" of an ideal I.

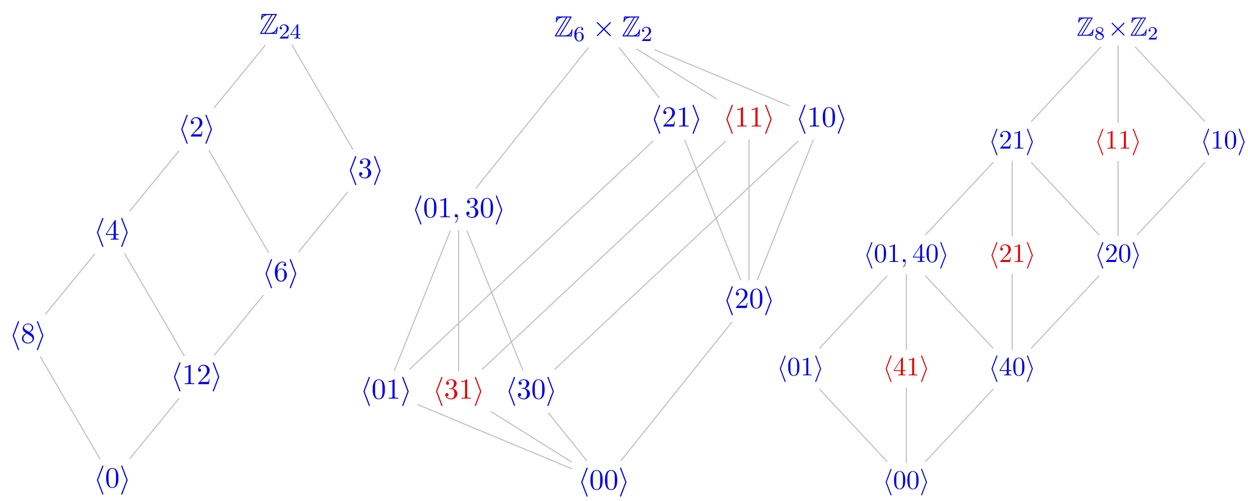
   (a) The *radical* of $I \subseteq R$ is the set

   $$\sqrt{I} := \big\{x \in R \mid x^n \in I \text{ for some } n \in \mathbb{N}\big\},$$

   and I is a *radical ideal* if $\sqrt{I} = I$.

   (i) Show that $\sqrt{I}$ is an ideal containing I.

   (ii) Find the radicals of all ideals of the rings $\mathbb{Z}_6 \times \mathbb{Z}_4$, $\mathbb{Z}_8 \times \mathbb{Z}_2$, $\mathbb{Z}_6 \times \mathbb{Z}_2$, and $\mathbb{Z}_{24}$. Denote these on the subring lattices by drawing an arrow from each I to $\sqrt{I}$. Three of these are shown below; see HW 11 for the fourth.



   (b) An element $a \in R$ is *nilpotent* if $a^n = 0$ for some $n \geq 1$. The *nilradical* of R is $\mathrm{Nil}(R) := \sqrt{0}$, the set of nilpotent elements. In class, we showed that this is the intersection of all prime ideals of R.

   (i) If $u \in R$ is a unit and $a \in R$ is nilpotent, show that $u + a$ is a unit.

   (ii) Show that $R/\mathrm{Nil}(R)$ has no nonzero nilpotent elements.

   (iii) Show that $\mathrm{Nil}(R/I) = \sqrt{I}/I$.

   (iv) Show that $\sqrt{I}$ is the intersection of all *prime ideals* that contain I.

2. The *Jacobson radical* of an ideal I, denoted $\mathrm{jac}(I)$, is the intersection of all *maximal ideals* that contain I. The *Jacobson radical of* R is $\mathrm{Jac}(R) := \mathrm{jac}(0)$, the intersection of all maximal ideals of R.

   (a) Show that $\sqrt{I} \subseteq \mathrm{jac}(I)$ and $\mathrm{Nil}(R) \subseteq \mathrm{Jac}(R)$.

   (b) Show that

   $$\mathrm{Jac}(R) = \big\{r \in R \mid 1 - rx \text{ is a unit for all } x \in R\big\}.$$

   (c) Determine $R/\mathrm{Jac}(R)$ for each of the rings $\mathbb{Z}_6 \times \mathbb{Z}_4$, $\mathbb{Z}_8 \times \mathbb{Z}_2$, $\mathbb{Z}_6 \times \mathbb{Z}_2$, and $\mathbb{Z}_{24}$.

3. Let R be a commutative ring with 1, and $D \subseteq R$ a multiplicatively closed subset containing no zero divisors. Consider the following set and equivalence relation $\sim$:

$$R \times D = \big\{(r,d) \mid r \in R,\ d \in D\big\}, \qquad (r_1,d_1) \sim (r_2,d_2) \ \Leftrightarrow\ r_1 d_2 = r_2 d_1.$$

   (a) Show that $\sim$ is an equivalence relation.

   (b) Let r/d denote the equivalence class containing (r,d), and the set of equivalence classes by $D^{-1}R$. Define addition and subtraction as follows:

$$\frac{r_1}{d_1} + \frac{r_2}{d_2} := \frac{r_1 d_2 + r_2 d_1}{d_1 d_2} \qquad \text{and} \qquad \frac{r_1}{d_1} \times \frac{r_2}{d_2} := \frac{r_1 r_2}{d_1 d_2}.$$

   Show that these operations are well-defined.

   (c) Show that the additive and multiplictive identities are $0/d$ and d/d, for any $d \in D$, and that the multiplicative inverse of r/d, if it exists, is $(r/d)^{-1} = d/r$.

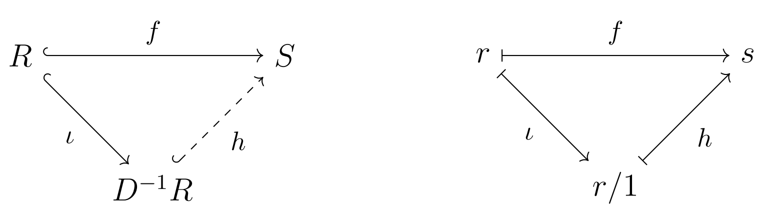
   (d) If $d \in D$, show that $\{rd/d \mid r \in R\}$ is a subring of $D^{-1}R$ and that

$$R \longrightarrow D^{-1}R, \qquad r \longmapsto rd/d$$

   is a injective homomorphism, therefy identifying R with a subring of $D^{-1}R$.

   (e) Under this identification, show that every $d \in D$ gets mapped to a unit in $D^{-1}R$.

   (f) Show that if $f \colon R \hookrightarrow S$ is an embedding to a commutative ring with 1, such that $f(d)$ is a unit, for every $d \in D$, then there is a unique ring homomorphism $h \colon D^{-1}R \hookrightarrow S$ such that $h \circ \iota = f$.



4. Let $\mathbb{F}$ be a field.

   (a) Show that there is a bijective correspondence between maximal ideals of $\mathbb{F}[x]$ and monic irreducible polynomials in $\mathbb{F}[x]$.

   (b) Show that if $M \subsetneq \mathbb{Z}[x]$ is a maximal ideal, then $M \bigcap \mathbb{Z} = (p)$ for some prime $p \neq 0$.

   (c) Show that there is a bijective correspondence between maximal ideals of $\mathbb{Z}[x]$ that contain p and monic irreducible polynomials in $\mathbb{Z}_p[x]$.

   (d) Characterize all maximal ideals of $\mathbb{Z}[x]$.