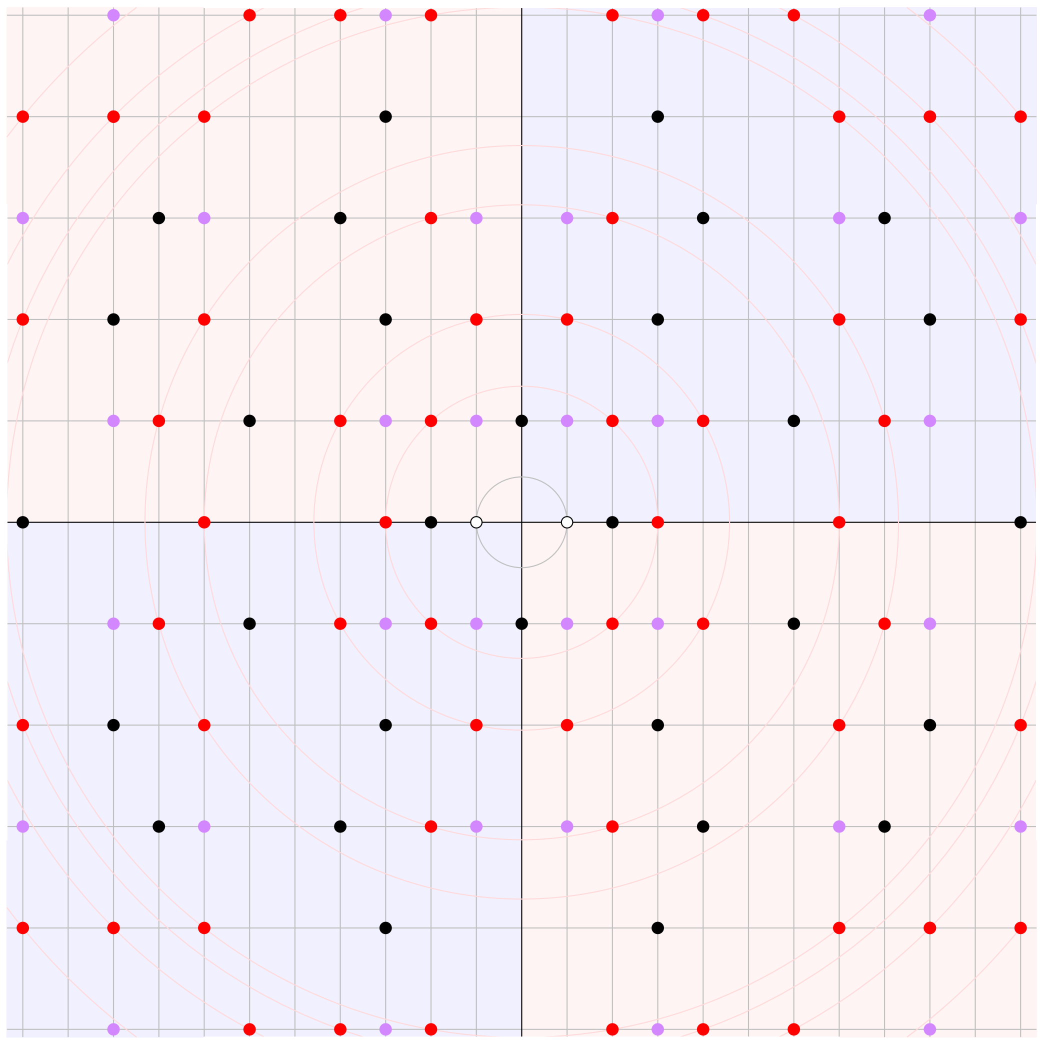
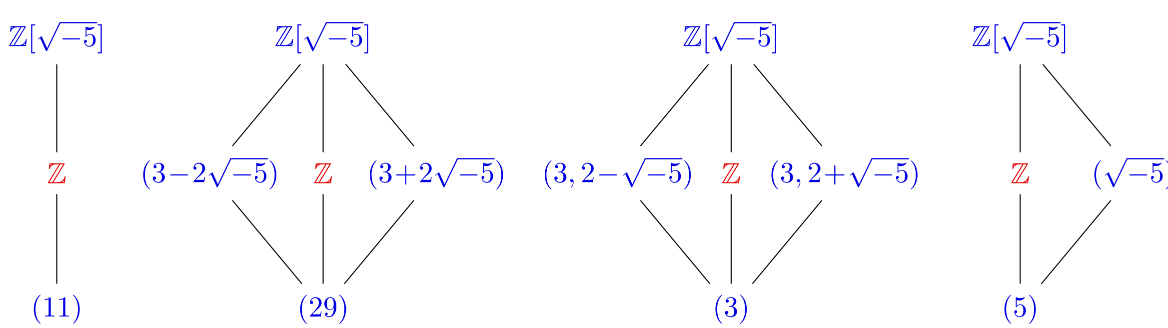
1. A picture illustrating the quadratic integers $R_{-5} = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$ as a subring of $\mathbb{C}$ is shown below, with the primes in black, and non-prime irreducibles colored.



   (a) Create an analogous picture for the ring $R_{-6} = \mathbb{Z}[\sqrt{-6}]$. First make a blank diagram with the norms of the quadratic integers labeled at each correponding lattice point.

   (b) Find primes $p \in \mathbb{Z}$ that in R_{-6} are inert, split (both reducible and irreducible), and ramified. Illustrate this with subring lattices. Examples for R_{-5} are shown below.



   (c) Give an elementary characterization of non-prime irreducibles in R_{-6}.

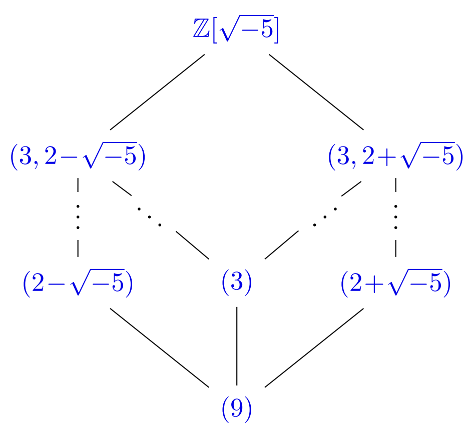
2. Prove that if $m = -3$, -7, or -11, then R_m is Euclidean with $d(r) = |N(r)|$ for all nonzero $r \in R_m$. [*Hint*: Mimic the proof of the same result for $m = -2$, -1, 2, and 3, but choose $d \in \mathbb{Z}$ nearest to $2t$ and then $c \in \mathbb{Z}$ so that c is as near to $2s$ as possible with $c \equiv d \pmod 2$, then set $q = (c + d\sqrt{m})/2$.]

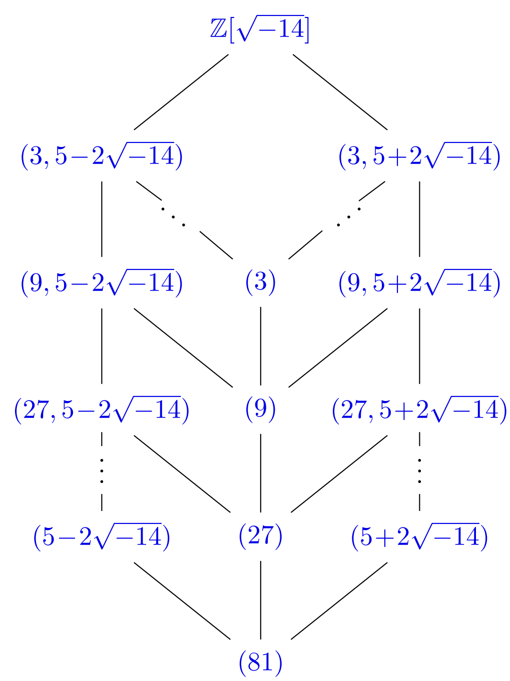
3. Suppose $P \neq 0$ is a prime ideal in the ring R_m of quadratic integers.
   (a) Show that $P \cap \mathbb{Z}$ is a prime ideal in $\mathbb{Z}$, so $P \cap \mathbb{Z} = (p)$ for some prime p in $\mathbb{Z}$.
   (b) Set $I = pR_m \subseteq P$ and form the quotient ring R/I. Show that R/I, as an additive group, is generated by two elements of finite order; hence R/I is finite.
   (c) Show that there is an epimorphism $R/I \twoheadrightarrow R/P$ and conclude that R/P is finite.
   (d) Conclude that every prime ideal in R_m is maximal.

4. The *class group* of the quadratic integer ring R_m measures the extent to which unique factorization fails. Two examples of this,

$$3^2 = (2+\sqrt{-5})(2-\sqrt{-5}) \in \mathbb{Z}[\sqrt{-5}], \qquad 3^4 = (5+2\sqrt{-14})(5-2\sqrt{-14}) \in \mathbb{Z}[\sqrt{-14}],$$

can be visualized in the lattices of ideas as follows:



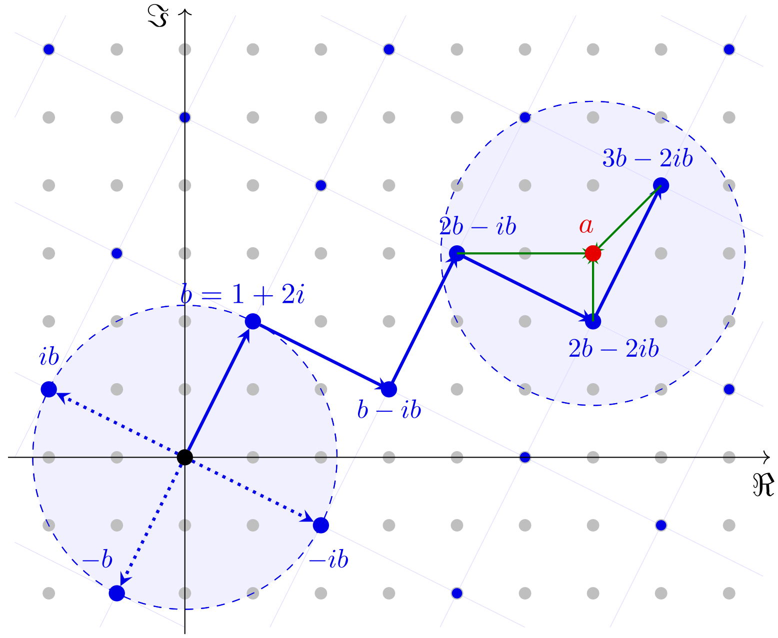


The class groups of these rings are $\mathrm{Cl}(\mathbb{Z}[\sqrt{-5}]) \cong C_2$ and $\mathrm{Cl}(\mathbb{Z}[\sqrt{-14}]) \cong C_4$, respectively. Create an analogous lattice for the ring $R_{-23} = \mathbb{Z}\big[\frac{1+\sqrt{-23}}{2}\big]$, whose class group is C_3. It is helpful to note the following product of ideals:

$$(2,\omega)(2,\bar{\omega}) = (2), \qquad \text{and} \qquad (2,\omega)^3 = (2-\omega), \qquad \text{where } w = \tfrac{-1+\sqrt{-23}}{2}.$$

5. If the quadratic integer ring R_m is a Euclidean domain, then for every nonzero $a, b \in R$, the division algorithm can be used to find $r, q \in R$ such that $a = bq + r$, $0 \leq N(r) < N(b)$.

   (a) The following visual shows all three ways to write $a = bq + r$ in the Gaussian integers, for $a = 6 + 3i$ and $b = 1 + 2i$. Carry out the division algorithm for $a = 9 + 8i$ and $b = 3 + i$, and create an analogous picture to illustrate it.



   (b) The ring $R_{-5} = \mathbb{Z}[\sqrt{-5}]$ is *not* Euclidean, and the following visual demonsrates how the division algorithm fails for $a = 5$ and $b = 2+\sqrt{-5}$. Find a and b in $R_{-6} = \mathbb{Z}[\sqrt{-6}]$ that confirms that it too is not Euclidean, and construct an analogous visual.