1. An element $e \in R$ in a commutative ring is called an *idempotent* if $e^2 = e$, and two nonzero idempotents e_1, e_2 are called an *orthogonal pair* if $e_1 + e_2 = 1$ and $e_1 e_2 = 0$. Show that the following are equivalent:

   (i) R contains an idempotent different from 0 and 1.

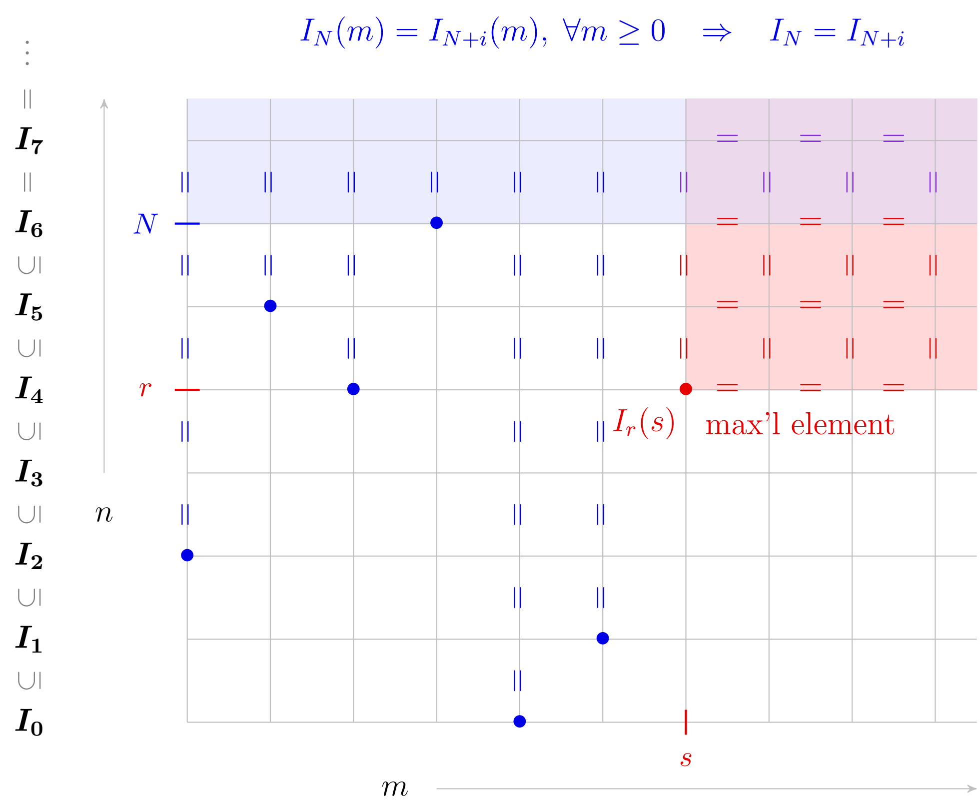
   (ii) R contains an orthogonal pair of idempotents.

   (iii) $R \cong R_1 \times R_2$ for some rings R_1 and R_2.

2. Let I be an ideal of $R[x]$, where R is an integral domain, and define

$$I(m) = \{a_m \mid f(x) = a_m x^m + \cdots + a_1 x + a_0 \in I\} \cup \{0\} \trianglelefteq R.$$

   In this problem, you will verify a few lemmas needed to establish Hilbert's basis theorem, that were left as exercises. The following picture summarizes our proof from class.



   (a) Prove the following, for all $m \in \mathbb{N}$.

      (i) $I(m)$ is an ideal of R.

      (ii) $I(m) \subseteq I(m+1)$.

      (iii) $I \subseteq J$, then $I(m) \subseteq J(m)$.

      Do any of these three results use the assumption that R has unity?

   (b) Modify our proof to establish Hilbert's basis theorem for formal power series. That is, show that if R is Noetherian, then $R[[x_1, \ldots, x_n]]$ is Noetherian as well.

3. Solutions to the following three systems are guaranteed by the Sunzi remainder theorem.

   (a) Solve the congruences

   $$x \equiv 1 \pmod{8}, \qquad x \equiv 3 \pmod{7}, \qquad x \equiv 9 \pmod{11}$$

   simultaneously for x in the ring $\mathbb{Z}$ of integers.

   (b) Solve the congruences

   $$x \equiv i \pmod{i+1}, \qquad x \equiv 1 \pmod{2-i}, \qquad x \equiv 1+i \pmod{3+4i}$$

   simultaneously for x in the ring $R_{-1} = \mathbb{Z}[i]$ of Gaussian integers.

   (c) Solve the congruences

   $$f(x) \equiv 1 \pmod{x-1}, \qquad f(x) \equiv x \pmod{x^2+1}, \qquad f(x) \equiv x^3 \pmod{x+1}$$

   simultaneously for $f(x)$ in $F[x]$, where F is a field in which $1+1 \neq 0$.

4. Consider the following rings R_i, for $i = 1, \dots, 6$, which are additionally $\mathbb{C}$-vector spaces:

   $R_1 = \mathbb{C}[x]/(x^3-1)$
   $R_2 = \mathbb{C} \times \mathbb{C} \times \mathbb{C}$
   $R_3 =$ the ring of upper triangular 2×2 matrices over $\mathbb{C}$
   $R_4 = \mathbb{C}[x]/(x-1) \times \mathbb{C}[x]/(x+i) \times \mathbb{C}[x]/(x-i)$
   $R_5 = \mathbb{C}[x]/(x^2+1) \times \mathbb{C}[x]/(x-1)$
   $R_6 = \mathbb{C}[x]/(x+1)^2 \times \mathbb{C}[x]/(x-1)$.

   (a) Compute the dimension of each R_i as a $\mathbb{C}$-vector space by giving an explicit basis.

   (b) Partition the rings $R_1, \dots, R_6$ into isomorphism classes.

5. For a fixed $a \in R$, denote the polynomial *evaluation map* by

   $$\phi_a \colon R[x] \longrightarrow R, \qquad \phi_a \colon f(x) \longmapsto f(a).$$

   (a) If $I = (3, x^2+1)$ in $\mathbb{Z}[x]$, show that $\phi_a(I) = \mathbb{Z}$.

   (b) Show by example how Part (c) can fail if 3 is replaced with a different odd prime p.

   (c) Given a polynomal, $f(x) \in R[x]$, substituting $a \in R$ for x determines a *polynomial function* $f \colon R \to R$, where $a \mapsto f(a)$. Show that if R is an infinite integral domain, then the mapping $f(x) \mapsto f$ assiging to each polynomial in $R[x]$ to its corresponding polynomial function is 1–1.