Chapter 7: Universal constructions

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Math 8510, Visual Algebra

Factoring maps

We've discussed a number of properties that can be described as

$$\text{"the minimal } \underline{\quad}\text{,"} \qquad \text{or} \qquad \text{"the maximal } \underline{\quad}\text{,"}$$

satisfying some condition.

We'll see to express this concisely in terms of maps and commutative diagrams.

This will highlight similarities and patterns that are inherent in seemingly different structures, streamline proofs, and lead to new insight.

Warm-up exercise (easy)

Given maps $g\colon G \to H$ and $h\colon H \to K$, their composition $f := h \circ g$ is a map from G to K.

I.e., there is *always* a map $f\colon G \to K$ making the following diagram commute:

$$
\begin{array}{ccc}
G & \overset{f = h \circ g}{\dashrightarrow} & K \\
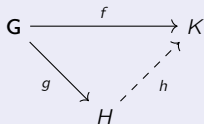& {\scriptstyle g}\searrow \quad \nearrow {\scriptstyle h} & \\
& H &
\end{array}
$$
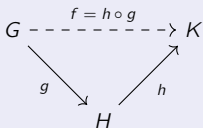
Factoring maps

Let's now consider two variants of the previous commutative diagram.

Definition

Given two maps...

1. *from the same domain*, $f\colon G \to K$, $g\colon G \to H$, when does there exist $h\colon H \to K$

2. *into the same codomain*, $f\colon G \to K$, $h\colon H \to K$, when does there exist $g\colon G \to H$

such that $f = h \circ g$?
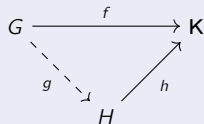
"f factors through g"   "f factors through h"

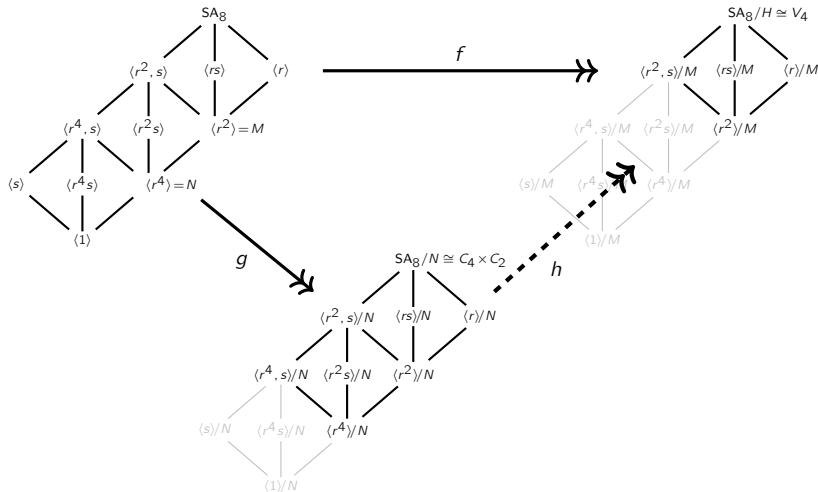We say that h and g are **factors** of f.

We'll do an example of each that will nicely illustrate when and why this happens.

Both will involve $G = \mathsf{SA}_8 = \langle r, s \rangle$, and its subgroups $N = \langle r^2 \rangle \cong C_4$, and $M = \langle r^4 \rangle \cong C_2$.

Factoring maps: a quotient between the codomains

Let $G = \mathsf{SA}_8 = \langle r, s \rangle$, and $N = \langle r^2 \rangle \cong C_4$, and $M = \langle r^4 \rangle \cong C_2$.
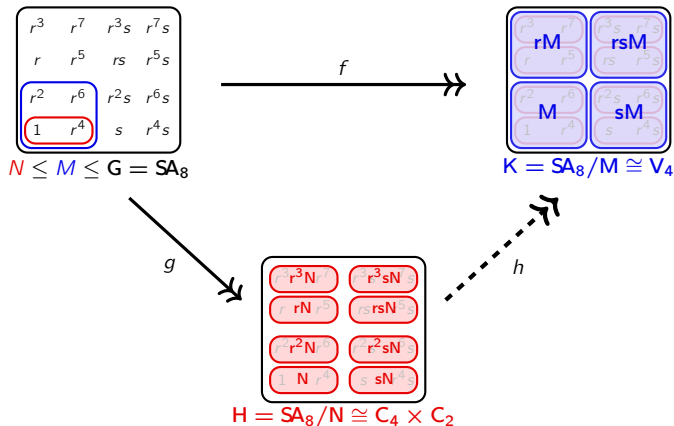
The standard quotient map $f \colon \mathsf{SA}_8 \to V_4$ can be factored:

Factoring maps: a quotient between the codomains

Formally, this map is defined by

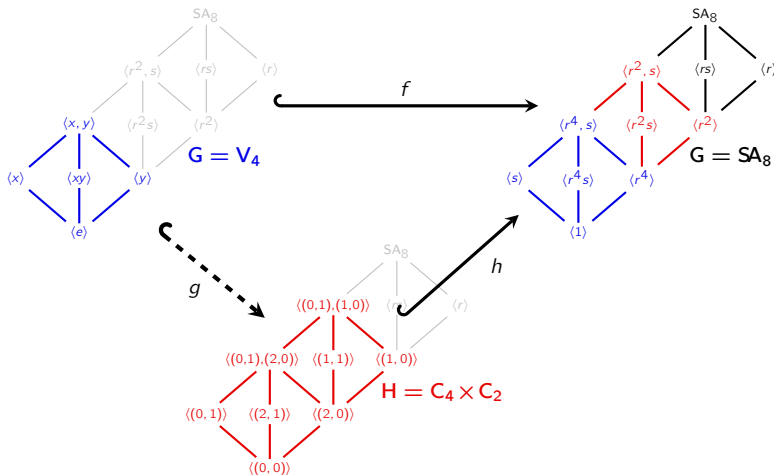$$h\colon \mathsf{SA}_8\,/N \longrightarrow \mathsf{SA}_8\,/M, \qquad h\colon gN \longmapsto gM.$$

Factoring maps: an embedding between the domains

Let $V_4 = \{e, x, y, xy\}$ and $\mathsf{SA}_8 = \langle r, s \rangle$. The embedding

$$f \colon V_4 \hookrightarrow \mathsf{SA}_8, \qquad x \longmapsto s, \quad y \longmapsto r^4$$

uniquely factors through $h \colon C_4 \times C_2 \to \mathsf{SA}_8$, where $(1, 0) \mapsto r^2$ and $(0, 1) \mapsto s$.
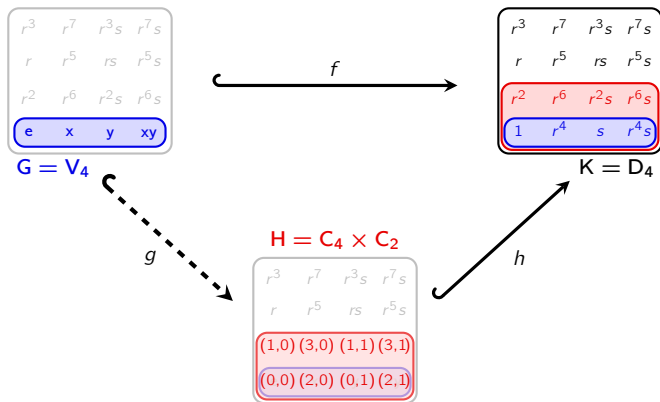
Factoring maps: an embedding between the domains

Let $V_4 = \{e, x, y, xy\}$ and $\mathsf{SA}_8 = \langle r, s \rangle$. Here's that same embedding

$$f \colon V_4 \hookrightarrow \mathsf{SA}_8, \qquad x \longmapsto r^4, \quad y \longmapsto s$$

that uniquely factors through $h \colon C_4 \times C_2 \to \mathsf{SA}_8$, where $(1,0) \mapsto r^2$ and $(0,1) \mapsto s$.
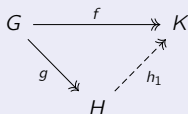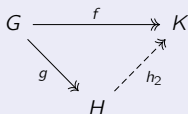
Canceling maps: when does existence imply uniqueness?

Proposition ("cancelation laws")

Suppose we have functions $g_i \colon G \to H$ and $h_i \colon H \to K$ between sets, for $i = 1, 2$.

If g is surjective, then it right-cancels: $\quad h_1 \circ g = h_2 \circ g \quad \Longrightarrow \quad h_1 = h_2$.

$$\Longrightarrow \quad h_1 = h_2.$$

If h is injective, then it left-cancels: $\quad h \circ g_1 = h \circ g_2 \quad \Longrightarrow \quad g_1 = g_2$.
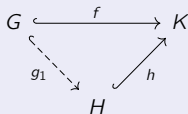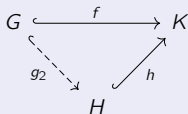
$$\Longrightarrow \quad g_1 = g_2.$$

Key idea

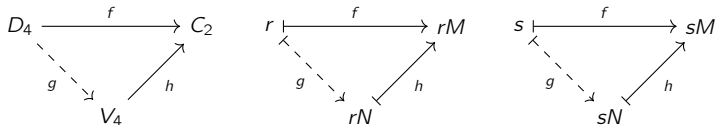Injective functions have left inverses; surjective functions have right inverses.

Failure of uniqueness: a quotient between domains

Let $D_4 = \langle r, s \rangle$ with subgroups $N = \langle r^2 \rangle \cong C_2$ and $M = \langle r \rangle \cong C_4$.

Their quotients are $V_4 \cong D_4/N = \{N, rN, sN, rsN\}$ and $C_2 \cong D_4/M = \{M, sM\}$.

Define the functions $f \colon D_4 \to C_2$ and $h \colon V_4 \to C_2$ as follows:

We must have $g \colon 1 \mapsto N$. Since $f \colon r \mapsto M$, then $g(r) \in \mathrm{Ker}(h) = \{rN, N\}$.

If $g(r) = N$, then g is not surjective, but we still have $f = h \circ g$.
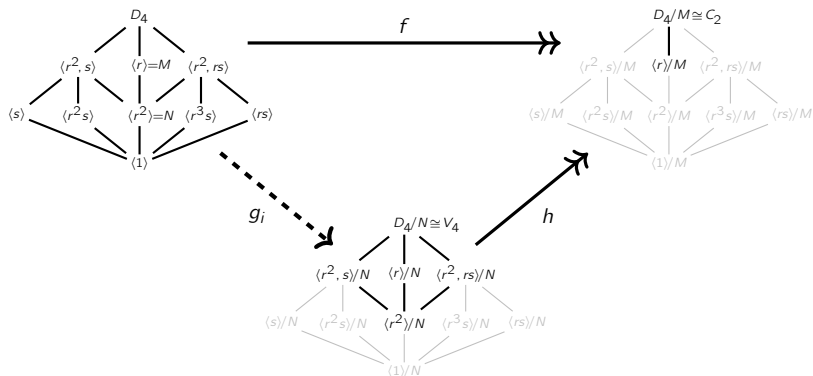
Warning!

The homomorphism $g \colon D_4 \to V_4$ is *not* uniquely defined! ($r \mapsto N$ would work too)

Moral: commutative diagrams can be deceiving!

Failure of uniqueness: a quotient between domains

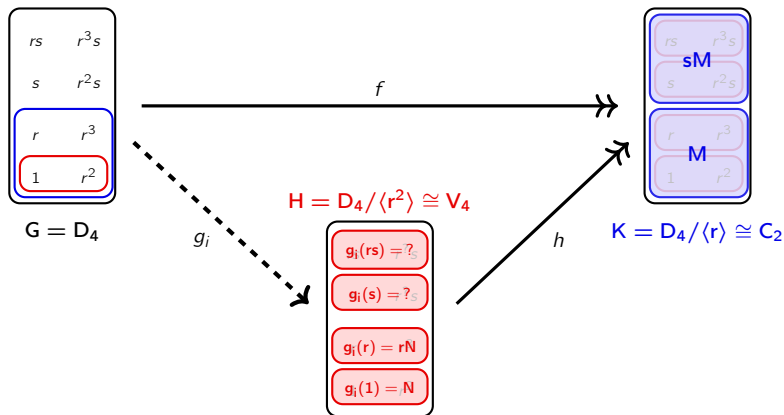Multiple maps g_i make this diagram commute; both $r \mapsto rN$ and $r \mapsto N$ work.

For surjective maps, $h \circ g_1 = h \circ g_2 \;\; \not\Rightarrow \;\; g_1 = g_2$.

Failure of uniqueness: a quotient between domains

Note that $g_i \colon D_4 \to V_4$ need not be surjective for the following diagram to commute.

Any choice of $g_i(r) \in \{N, rN\}$ and $g_i(s) \in \{sN, rsN\}$ would work.

Failure of uniqueness: an embedding between codomains

Consider two maps from $G = C_2 = \{1, a\}$ into $H = V_4 = \{e, x, y, xy\}$ and $K = D_4 = \langle r, s \rangle$:

$$f \colon C_2 \longrightarrow D_4, \quad f(a) = r^2, \qquad\qquad g \colon C_2 \longrightarrow V_4, \quad f(a) = x.$$

There are multiple embeddings $h_i \colon V_4 \hookrightarrow D_4$ that make this diagram commute:
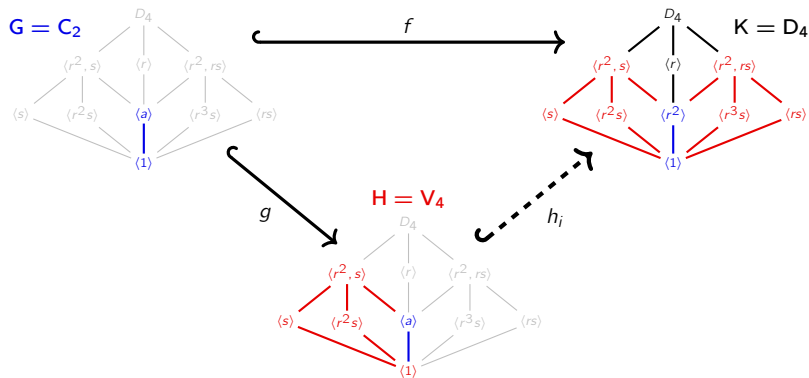
For injective maps, $h_1 \circ g = h_2 \circ g \;\;\not\Rightarrow\;\; h_1 = h_2$.

Failure of uniqueness: an embedding between codomains

Here is another way to see why there are multiple embeddings $h_i \colon V_4 \hookrightarrow D_4$ that make this diagram commute:
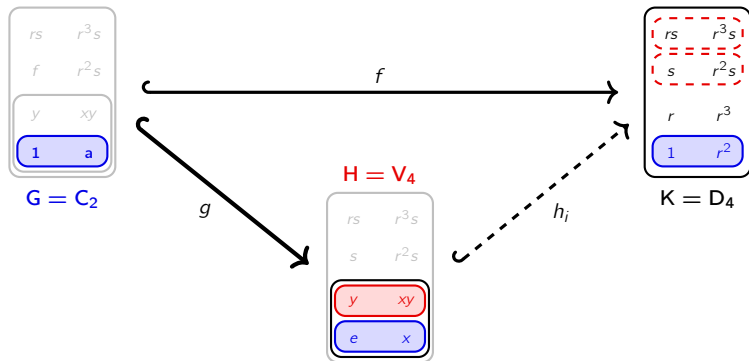

For injective maps, $h_1 \circ g = h_2 \circ g \;\not\Rightarrow\; h_1 = h_2$.

Factoring non-homomorphisms

> **Definition**
>
> Let G/N be a set (not necessarily a group) of equivalence classes. The map ϕ from G
> **descends to a map from** G/N if it factors through the canonical quotient $\pi\colon G \to G/N$.

For example, we have seen that:

- the map $\phi\colon G \to \mathrm{cl}_G(H)$ **descends to a bijection** $G/N_G(H) \to \mathrm{cl}_G(H)$.
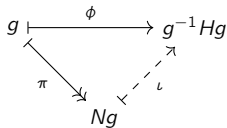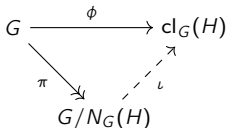
- the map $\phi\colon G \to \mathrm{cl}_G(g)$ **descends to a bijection** $G/C_G(g) \to \mathrm{cl}_G(g)$.
- For a fixed $s \in S$, $\phi\colon G \to \mathrm{orb}(s)$ **descends to a bijection** $G/\mathrm{stab}(s) \to \mathrm{orb}(s)$.
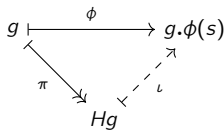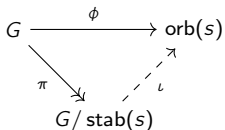
Motivating the co-universal property of quotient groups

Definition

Given $H \leq G$, the **canonical inclusion map** is

$$\iota \colon H \hookrightarrow G, \qquad \iota \colon h \longmapsto h.$$

If $H \trianglelefteq G$, the **canonical quotient map** is

$$\pi \colon G \longrightarrow G/H, \qquad \pi \colon g \longmapsto gH.$$

There does not exist a homomorphism $\phi \colon \mathbb{Z}_3 \to \mathbb{Z}_4$ with $\phi(1) = 1$. To formalize this:

the canonical quotient $f \colon \mathbb{Z} \twoheadrightarrow \mathbb{Z}_4$ does not factor through $g \colon \mathbb{Z} \twoheadrightarrow \mathbb{Z}_3$.

That is, there does *not* exist $\phi \colon \mathbb{Z}_3 \to \mathbb{Z}_4$ making this diagram commute:

Preview: such a map exists iff $\mathsf{Ker}(\pi) \leq \mathsf{Ker}(f)$, i.e., f collapses at least as much as π.

Motivating the co-universal property of quotient groups

Does $\phi\colon \mathbb{Z}_8 \to \mathbb{Z}_{12}$, where $\phi(1) = 3$, define a homomorphism?

Is there a homomorphism ϕ making the following diagram commute?
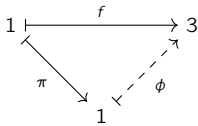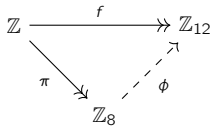
Note that $\mathrm{Ker}(f) = 4\mathbb{Z}$ is a subgroup of $\mathrm{Ker}(\pi) = 8\mathbb{Z}$, and so f factors through π.

Not only does ϕ exist, it is automatically unique by the cancellation laws.

The co-universal property of quotient groups

Theorem

Let $N \trianglelefteq G$ and $f \colon G \to K$ be a homomorphism such that $N \leq \mathsf{Ker}(f)$. Then

1. f uniquely factors through $\pi \colon G \to G/N$ (i.e., $\exists! h \colon G/N \to K$ such that $g = h \circ \pi$).
2. h is injective iff $\mathsf{Ker}(f) = N$.

Proof (i)

Assume WLOG that f is onto (otherwise, take $K = \mathsf{Im}(f)$). Define $h \colon G/N \to H$ by

Well-defined: If $xN = yN$, then $y^{-1}xN = N$, so $y^{-1}x \in N = \mathsf{Ker}(\pi) \leq \mathsf{Ker}(f)$. Now,

$$f(y^{-1}x) = 1 \quad \implies \quad f(y)^{-1}f(x) = 1 \quad \implies \quad h(xN) = f(x) = f(y) = h(yN). \qquad \checkmark$$

Homomorphism: $h(xNyN) = h(xyN) = f(xy) = f(x)f(y) = h(xH)h(yN).$ $\qquad\qquad\qquad \checkmark$

Uniqueness: Follows from existence, since f and π are quotients (cancellation laws). $\qquad \checkmark$
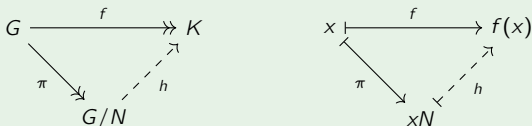
The co-universal property of quotient groups

Theorem

Let $N \trianglelefteq G$ and $f \colon G \to K$ be a homomorphism such that $N \leq \mathsf{Ker}(f)$. Then

1. f uniquely factors through $\pi \colon G \to G/N$ (i.e., $\exists! h \colon G/N \to K$ such that $g = h \circ \pi$).
2. h is injective iff $\mathsf{Ker}(f) = N$.

Proof (ii)

Assume WLOG that f is onto. We just found the unique h such that
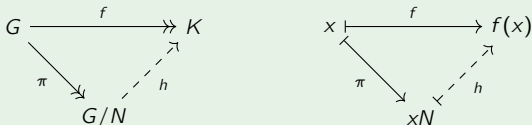
Let $H = \mathsf{Ker}(f)$, and note that

$$\mathsf{Ker}(h) = \big\{ xN \mid f(x) = 1_K \big\} = \big\{ xN \mid x \in H \big\} = H/N.$$

Note that h is injective iff $|\mathsf{Ker}(h)| = 1$, or equivalently, $H = N$. $\qquad\square$

Co-universal property of quotient groups $\Rightarrow$ FHT

Corollary: Fundamental homomorphism theorem

If $f\colon G \to H$ is a homomorphism, then $G/\operatorname{Ker}(f) \cong \operatorname{Im}(f)$.

Proof

Let $K = \operatorname{Im}(f)$ and $N = \operatorname{Ker}(f)$ with canonical quotient map $\pi\colon G \to G/N$.

By construction, $\operatorname{Ker}(f) = N = \operatorname{Ker}(\pi)$.

By the co-universal property of quotient maps, f factors through the quotient:

$$
\begin{array}{ccc}
G & \xrightarrow{\;\;f\;\;} & \operatorname{Im}(f) \\
\ \ \downarrow{\scriptstyle \pi} & \nearrow{\scriptstyle \iota} & \\
G/N & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
x & \xmapsto{\;\;f\;\;} & f(x) \\
\ \ \downarrow{\scriptstyle \pi} & \nearrow{\scriptstyle \iota} & \\
xN & &
\end{array}
$$

Since $\operatorname{Ker}(f) = N$, the map ι is injective by Part (ii) of the previous theorem.

Therefore, ι is an isomorphism. $\qquad\qquad\square$

Abstracting the (co)-universal property

To motivate where we're going, let's rephrase what we just did as

> "G/N is the *largest quotient that collapses N*, in that any other homomorphism collapsing N *factors through* $\pi\colon G \to G/N$ *uniquely*."
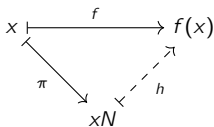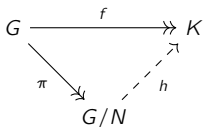
Compare this to what we know about the commutator subgroup G':

> "G/G' is the *largest abelian quotient* of G, in that any other homomorphism to an abelian group *factors through* $\alpha\colon G \to G/G'$ *uniquely*."
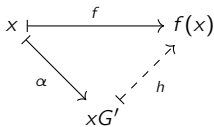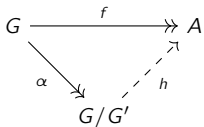
Abstracting the (co)-universal property

The **co-universal property** of quotients came with a distinguished (maximal)

- group G/N, and
- canonical map $\pi \colon G \to G/N$.

Definition

A **co-universal pair** (C, χ) for a group G w.r.t. a property consists of:

- a group C, with
- an incoming map $\chi \colon G \to C$,

such that every $f \colon G \to H$ with the same property factors through χ uniquely.

I.e., there is a unique homomorphism $h \colon C \to H$ between co-domains such that $f = h \circ \chi$.

$$
\begin{array}{ccc}
G & \xrightarrow{\quad f \quad} & H \\
& \chi \searrow & \nearrow h \\
& & C
\end{array}
$$
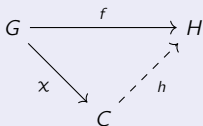
Abstracting the (co)-universal property

Proposition

If G has a co-universal pair (C, χ) w.r.t. some property, C is unique up to isomorphism.

Proof

Let (C, χ) and (C', χ') be co-universal. Start with (C, χ), and take $H = C'$ and $f = \chi'$.

By definition, $\exists! h \colon C \to C'$ such that $\chi' = h \circ \chi$. Reverse the roles, and we get:
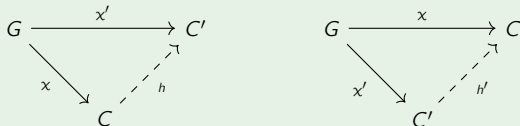
We can "stack" one diagrams on the other, and vice-versa:
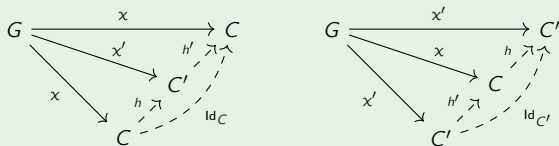
By uniqueness, $h \circ h' = \mathsf{Id}_C$ (left), and $h' \circ h = \mathsf{Id}_{C'}$ (right). Thus, $C \cong C'$. $\qquad\square$

A co-universal property and nilpotency

Recall that we characterized nilpotent groups via iterative "maximal central descents."

Given $N \trianglelefteq G$, the maximal central descent $[G, N]$ is characterized as being

"*the smallest subgroup L such that N/L is central in G/L*".

We can phrase this as a co-universal property.

Consider (L, λ), where $L = [G, N]$ and $\lambda \colon G \to G/L$ is the canonical quotient.

Co-universal property of central descents (HW)

Let $N \trianglelefteq G$ and $f \colon G \to K$ for which $f(N)$ is central. Then f uniquely factors through the canonical quotient map $\lambda \colon G \to G/L$, where $L = [G, N]$.

That is, there is a unique homomorphism $h \colon G/L \to K$ for which $f = h \circ \lambda$.
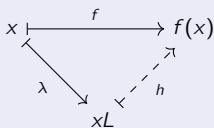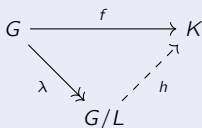
Universal vs. co-universal properties

We call the examples we've seen co-universal because the map is between the co-domains.

The "dual" version, where the maps is between the domains, are universal properties.

Most books don't distinguish these two, and use "universal" for both.

The examples we've seen were maximal quotients. Let's now look at maximal subgroups.

Universal property of centers

Let $H \leq G$ for which $xz = zx$ for all $z \in H$ and $x \in G$. The canonical inclusion $g \colon H \hookrightarrow G$ uniquely factors through $\zeta \colon Z(G) \hookrightarrow G$.

That is, there is a unique embedding $g \colon H \hookrightarrow Z(G)$ for which $f = \zeta \circ g$.
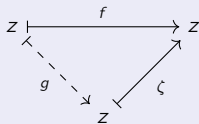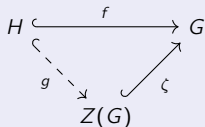
Another universal property

Universal property of central ascents

Given $N \trianglelefteq G$, suppose that $H/N \leq Z(G/N)$. The canonical inclusion $H/N \hookrightarrow G/N$ uniquely factors through $\zeta \colon Z(G/N) \hookrightarrow G/N$.

That is, there is a unique embedding $g \colon H/N \hookrightarrow Z(G/N)$ for which $f = \zeta \circ g$.
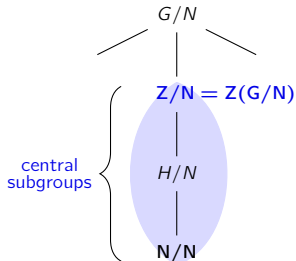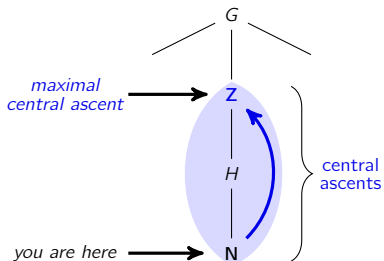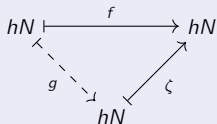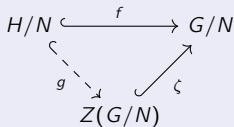
Universal pairs and universal constructions

Definition

A **universal pair** (U, υ) for G w.r.t. a property consists of a group U and map $\upsilon \colon U \to G$, such that every other $f \colon H \to G$ with the same property factors through υ uniquely.

That is, $\exists! g \colon H \to U$ between the *domains* such that $f = \upsilon \circ g$.

$$
\begin{array}{ccc}
H & \xrightarrow{\quad f \quad} & G \\
& {\scriptstyle g} \searrow \quad \nearrow {\scriptstyle \upsilon} & \\
& U &
\end{array}
$$
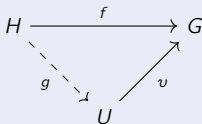
Proposition (HW)

If G has a universal pair (U, υ) w.r.t. some property, then U is unique up to isomorphism.

It's not standard or necessary to characterize a simple concept like $Z(G)$ with a universal property. We did it as a "warm up."

Soon, we'll *define* concepts by a (co-)universal property.

These are examples of universal constructions.

Motivation: direct product vs. direct sums

> **Open-ended question**
>
> *What is the limit of $\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}\}$, as $n \to \infty$?*

- Define $\mathbb{R}^\infty$ to be the space of all infinite sequences

$$\mathbb{R}^\infty := \prod_{i=1}^{\infty} \mathbb{R} := \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots = \{(a_1, a_2, a_3, \ldots) \mid a_i \in \mathbb{R}\}.$$

This space contains "vectors" such as $(1, 1, 1, \ldots)$. We'll call it the "**direct product**."

- Define $\mathbb{E}^\infty$ to be the space of all finite sums, like

$$\mathbf{e} = a_1 \mathbf{e}_1 + \cdots + a_n \mathbf{e}_n = \sum_{i=1}^{n} a_i \mathbf{e}_i, \qquad ||\mathbf{v}|| = \sqrt{a_1^2 + \cdots + a_n^2}.$$

We'll call this the "**direct sum**".

$$\mathbb{E}^\infty := \bigoplus_{i=1}^{\infty} \mathbb{R}\mathbf{e}_i := \mathbb{R}\mathbf{e}_1 \oplus \mathbb{R}\mathbf{e}_2 \oplus \mathbb{R}\mathbf{e}_3 \oplus \cdots = \left\{\sum_{i=1}^{k} a_i \mathbf{e}_i \mid a_i \in \mathbb{R}, \ k \geq 1\right\}$$

$$\cong \{(a_1, a_2, a_3, \ldots) \mid a_i \in \mathbb{R}, \text{ all but finitely many } a_j \text{ are zero}\}.$$

Motivation: direct product vs. direct sums

Define the canonical quotient maps for each $i = 1, 2, \dots$ as

$$\pi_i \colon \mathbb{R} \times \mathbb{R} \times \cdots \longrightarrow \mathbb{R}, \qquad \pi_i \colon (a_1, a_2, \dots) \longmapsto a_i.$$

The direct product is the "*smallest P that projects onto each factor.*"

Given any family $f_i \colon X \to \mathbb{R}$ of maps, each f_i factors through the projection $\pi_i \colon P \to \mathbb{R}$.
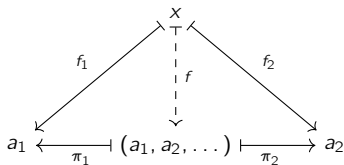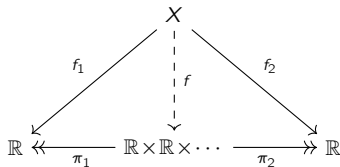
Let's see why this fails if we tried to use $\mathbb{E}^\infty$ for P:
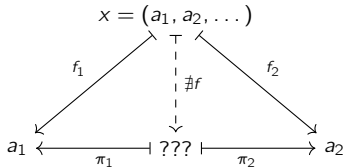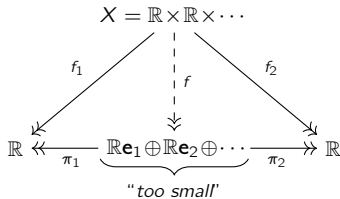
Motivation: direct product vs. direct sums

Define the natural inclusion map for each $j = 1, 2, \dots$ as

$$\iota_j \colon \mathbb{R}\mathbf{e}_j \hookrightarrow \bigoplus_{i=1}^{\infty} \mathbb{R}\mathbf{e}_i, \qquad \iota_j \colon a_j \mathbf{e}_j \longmapsto a_j \mathbf{e}_j$$

The direct sum is the "*smallest S that each factor embeds into.*"

Given any family $f_j \colon \mathbb{R}\mathbf{e}_j \to X$ of maps, each ι_j factors through the embedding $\iota_j \colon \mathbb{R}\mathbf{e}_j \hookrightarrow S$.
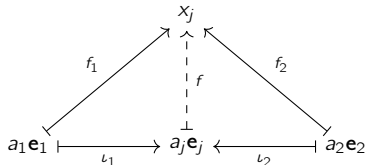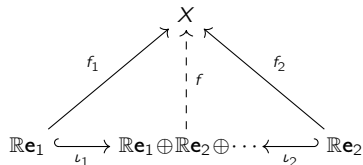
Let's see why this fails if we try to use $\mathbb{R}^{\infty}$ for S:
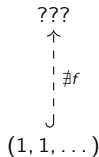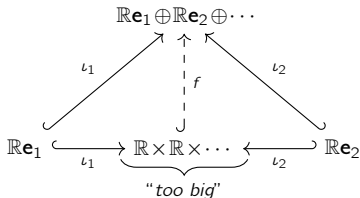
Returning to groups

Let $\{G_\alpha \mid \alpha \in A\}$ be a nonempty family of groups. We will define their product and co-product via a universal construction.
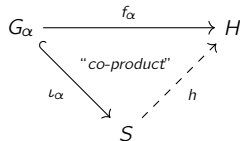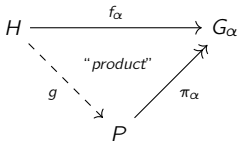
Remark

Existence of the map needed to make these diagrams commute does *not* imply uniqueness from the cancellation laws – each is the "wrong type" of diagram for that.

The fact that there are such groups that guarantee uniqueness indicates that the definitions are capturing something fundamentally important.

Definition

The **product** of $\{G_\alpha \mid \alpha \in A\}$ is a group P with a family of homomorphisms $\{\pi_\alpha \colon P \to G_\alpha \mid \alpha \in A\}$, satisfying:

> Given any group H and homomorphisms $f_\alpha \colon H \to G_\alpha$, there is a unique homomorphism $g \colon H \to P$ such that $\pi_\alpha \circ g = f_\alpha$ for all $\alpha \in A$.

Products: surjectivity and uniqueness

Proposition

If $\{G_\alpha \mid \alpha \in A\}$ has a product, it is unique up to isomorphism, and each π_α is surjective.

Proof

We've shown uniqueness.

To show that π_α is surjective, consider $\pi_\beta \colon P \to G_\beta$, and take $H = G_\alpha$.

Define f_β to be the identity map if $\beta = \alpha$ and the trivial map otherwise. That is,

$$f_\alpha \colon G_\alpha \longrightarrow G_\beta, \qquad f_\alpha(x) = \begin{cases} x, & \alpha = \beta \\ 1, & \alpha \neq \beta. \end{cases}$$

Every element $x \in G_\beta$ has a π_β-preimage, $g(x) \in P$.
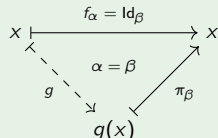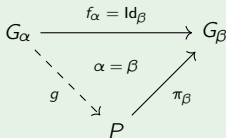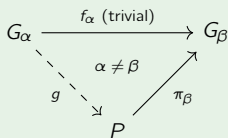
Products: existence

Proposition

The product of $\{G_\alpha \mid \alpha \in A\}$ is the Cartesian product, $P = \prod_{\alpha \in A} G_\alpha$.

Proof

Define the canonical projection maps as

$$\pi_\beta \colon P \longrightarrow G_\beta, \qquad \pi_\beta \colon (x_\alpha)_{\alpha \in A} \longmapsto x_\beta.$$

Suppose we have another family of maps $f_\alpha \colon H \to G_\alpha$, for each $\alpha \in A$.

Goal. Show $\exists! g \colon H \to P$ such that $f_\alpha = \pi_\alpha \circ g$ for all $\alpha \in A$.

$$
\begin{array}{ccc}
H & \xrightarrow{\ f_\beta\ } & G_\beta \\
& & \\
\searrow{\scriptstyle g} & \nearrow{\scriptstyle \pi_\beta} & \\
& \prod_{\alpha \in A} G_\alpha &
\end{array}
\qquad
\begin{array}{ccc}
x & \xmapsto{\ f_\beta\ } & f_\beta(x) \\
& & \\
\searrow{\scriptstyle g} & \nearrow{\scriptstyle \pi_\beta} & \\
& (f_\alpha(x))_{\alpha \in A} &
\end{array}
$$

Uniqueness. Suppose $\exists h \colon H \to P$ for which $f_\alpha = \pi_\alpha \circ h$ for all $\alpha \in A$.

This means $\pi_\alpha \circ g = \pi_\alpha \circ h$. Take $x \in H$, note that

$$h(x)_\beta = \pi_\beta(h(x)) = f_\beta(x) = \pi_\beta(g(x)) = g(x)_\beta. \qquad \square$$

Co-products

Definition

The **co-product** of $\{G_\alpha \mid \alpha \in A\}$ is a group S with a family of homomorphisms $\{\iota_\alpha \colon G_\alpha \to S \mid \alpha \in A\}$, satisfying:

 Given any group H and homomorphisms $f_\alpha \colon G_\alpha \to H$, there is a unique homomorphism $h \colon S \to H$ such that $h \circ \iota_\alpha = f_\alpha$ for all $\alpha \in A$.
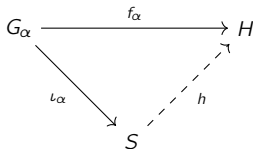
Exercise (HW)

If $\{G_\alpha \mid \alpha \in A\}$ has a co-product, it is unique up to isomorphism, and each ι_α is injective.

Showing existence of a co-product is trickier – it a construction that we have not yet seen.

The product of C_2 and C_2 has order 4. The co-product is infinite.

Categories

Some constructions we've recently seen have analogues for other mathematical objects.

We can define the product and coproduct of sets, topological spaces, rings, vector spaces, etc.

Many structural results carry over, so we'd like to generalize these in a common framework.

The mathematical field that addresses these questions is called **category theory**.

Definition

A **category** $\mathcal{C}$ consists of

- a class $\mathrm{Ob}(\mathcal{C})$ of **objects**,
- a class $\mathrm{Hom}(\mathcal{C})$ of **morphisms** between objects, with identities, closure, and associativity.

- Examples of "objects" include sets, groups, rings, vector spaces, topological spaces, etc.,
- "Morphisms" are meant to be "structure-preserving maps."

Categories

Think of the category $\mathcal{C} = \mathbf{Grp}$ of groups as a massive directed multigraph, where

- each node represents a group
- there is a directed edge from A to B for each homomorphism $f : A \to B$.

We require: identity, composition, and associativity.
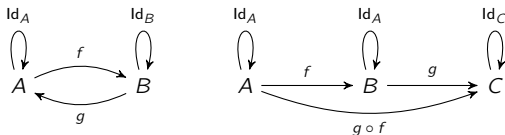
Denote the morphisms from A to B by $\text{Hom}_{\mathcal{C}}(A, B)$.

(i) Every group has an **identity morphism**: for every $A \in \text{Ob}(\mathcal{C})$, there is $\text{Id}_A \in \text{Hom}_{\mathcal{C}}(A, A)$ satisfying

$$f \circ \text{Id}_A = f, \ \text{ for all } f \in \text{Hom}_{\mathcal{C}}(A, B), \qquad \text{Id}_A \circ g = g, \ \text{ for all } h \in \text{Hom}_{\mathcal{C}}(B, A).$$

(ii) Morphisms are **closed under composition**:

$$\text{If } f \in \text{Hom}_{\mathcal{C}}(A, B) \ \text{ and } \ g \in \text{Hom}_{\mathcal{C}}(B, C), \ \text{ then } \ g \circ f \in \text{Hom}_{\mathcal{C}}(A, C).$$

(iii) Composition of morphisms is **associative**:

$$\text{If } f \in \text{Hom}_{\mathcal{C}}(A, B), \ g \in \text{Hom}_{\mathcal{C}}(B, C), \ h \in \text{Hom}_{\mathcal{C}}(C, D), \ \text{ then } h \circ (g \circ f) = (h \circ g) \circ f.$$

Abstracting the notion of "one-to-one" and "onto"

Definition

Let $f, f_1, f_2 \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g, g_1, g_2 \in \mathrm{Hom}_{\mathcal{C}}(B, C)$. Then

1. g is a **monomorphism** if $g \circ f_1 = g \circ f_2$ implies $f_1 = f_2$.
2. f is an **epimorphism** if $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$,

Sometimes, we'll say "*mono*" and "*epi*" (nouns) or *monic* and "*epic*" (adjectives).

A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is an **isomorphism** if it has a two-sided inverse.

That is, if $\exists g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that $g \circ f = \mathrm{Id}_A$ and $f \circ g = \mathrm{Id}_B$.

We say A and B are **equivalent**.

"*monomorphism*" (one-to-one)   "*epimorphism*" (onto)   "*equivalent*" (isomorphic)

Abstracting the notion of "product" and "coproduct"

Definition

Consider a category $\mathcal{C}$ and a non-empty collection $\{B_i \mid i \in I\}$ of objects.

A **product** for $\{B_i\}$ is $P \in \mathrm{Ob}(\mathcal{C})$ with a family $\{\pi_i \in \mathrm{Hom}(P, B_i) \mid i \in I\}$ such that:

> Given any $A \in \mathrm{Ob}(\mathcal{C})$ and $\{f_i \in \mathrm{Hom}_{\mathcal{C}}(A, B_i) \mid i \in I\}$, there is a unique $g \in \mathrm{Hom}_{\mathcal{C}}(A, P)$ such that $\pi_i \circ g = f_i$ for all $i \in I$.

A **coproduct** for $\{B_i\}$ is $S \in \mathrm{Ob}(\mathcal{C})$ with $\{\iota_i \in \mathrm{Hom}(A_i, S) \mid i \in I\}$ such that:

> Given any $B \in \mathrm{Ob}(\mathcal{C})$ and family $\{f_i \in \mathrm{Hom}_{\mathcal{C}}(A_i, B) \mid i \in I\}$, there is a unique $h \in \mathrm{Hom}_{\mathcal{C}}(S, B)$ such that $h \circ \iota_i = f_i$ for all $i \in I$.
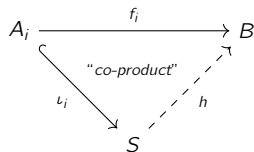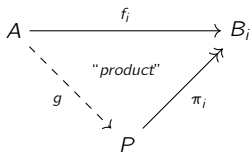
$$\tag{1}$$

It can be shown that the π_i's are epimorphisms, and ι_i's are monomorphisms.

A few counterintuitive facts

- *Isomorphisms $\neq$ mono + epi!* In the category **Rng**, the non-surjective morphism

$$g\colon \mathbb{Z} \longrightarrow \mathbb{Q}, \qquad g(n) = n$$

is both monic and epic.

$$R \underset{g_2}{\overset{g_1}{\rightrightarrows}} \mathbb{Z} \xrightarrow{\ f\ } \mathbb{Q} \qquad\qquad \mathbb{Z} \xrightarrow{\ g\ } \mathbb{Q} \underset{h_2}{\overset{h_1}{\rightrightarrows}} \mathbb{R}$$

The equality $f \circ g_1 = f \circ g_2$, implies $g_1 = g_2$, and $h_1 \circ g = h_2 \circ g$, forces $h_1 = h_2$.

However, g is not an isomorphism because it does not have a left or a right inverse.

- *The same concept across different categories can seem very different!*

Category	Objects	Morphisms	Product	coproduct
Set	sets	functions	Cartesian product	disjoint union
Grp	groups	homomorphisms	direct product	free product
Ab	abelian groups	homomorphisms	direct product	direct sum
Ring	rings w/ 1	ring homomorphisms	direct product	free product
Field	fields	field embeddings	none	none
Vect$_{\mathbb{F}}$	$\mathbb{F}$-vector spaces	linear functions	direct product	direct sum
Top	topological spaces	continuous maps	product topology	disjoint union

A functor from **Top** to **Grp**

Sometimes, there are structure-preserving maps between categories.
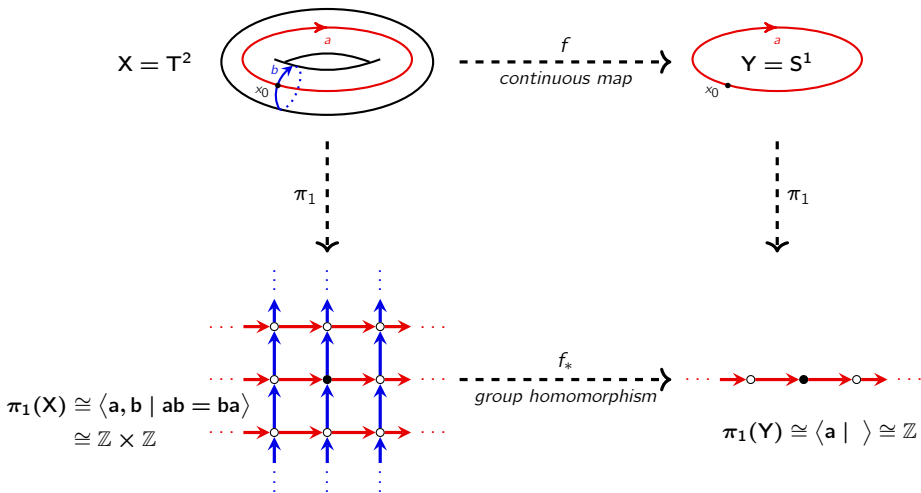
This is an example of a **functor**.

A functor from **Top** to **Grp**

The fundamental group of X is the group $\pi_1(X)$ of all "loops up to equivalence."

A continuous map $f \colon X \to Y$ induces a homomorphism

$$f_* \colon \pi_1(T^2) \longrightarrow \pi_1(S^1), \qquad f_* \colon (a, b) \longmapsto a.$$

Formally, π_1 is a functor from **Top$_\bullet$** to **Grp**, defined as:

$$\pi_1 \colon \mathrm{Ob}(\mathbf{Top_\bullet}) \longrightarrow \mathrm{Ob}(\mathbf{Grp}) \qquad\qquad \mathcal{F} \colon \mathrm{Hom}(\mathbf{Top_\bullet}) \longrightarrow \mathrm{Hom}(\mathbf{Grp})$$

$$X \longmapsto \pi_1(X) \qquad\qquad X \xrightarrow{f} Y \quad \longmapsto \quad \pi_1(X) \xrightarrow{f_*} \pi_1(Y)$$

For arbitrary (pointed) topological spaces (X, x_0) and (Y, y_0):

$$
\begin{array}{ccc}
X & \xrightarrow{\;f\;} & Y \\[2pt]
{\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_1} \\[4pt]
\pi_1(X) & \xrightarrow{\;f_*\;} & \pi_1(Y)
\end{array}
$$

f is a contin. b/w topological spaces, with $f(x_0) = y_0$

f_* is a homomorphism b/w fundamental groups

Covariant and contravariant functors

Definition

A (**covariant**) **functor** $\mathcal{F}$ from $\mathcal{C}$ to $\mathcal{D}$ is a function that sends

- objects A of $\mathcal{C}$ to objects $\mathcal{F}(A)$ of $\mathcal{D}$,
- morphisms $f\colon A \to B$ in $\mathcal{C}$ to morphisms $\mathcal{F}(f)\colon \mathcal{F}(A) \to \mathcal{F}(B)$ in D satisfying:
    - $\mathcal{F}(\mathsf{Id}_A) = \mathsf{Id}_{\mathcal{F}(A)}$ for all $A \in \mathsf{Ob}(\mathcal{C})$
    - $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$ for all morphisms $f\colon A \to B$ and $g\colon B \to C$.

$$A \xrightarrow{\;f\;} B \xrightarrow{\;g\;} C \qquad\qquad \mathcal{F}(A) \xrightarrow{\;\mathcal{F}(f)\;} \mathcal{F}(B) \xrightarrow{\;\mathcal{F}(g)\;} \mathcal{F}(C)$$
$$g \circ f \qquad\qquad\qquad\qquad\qquad \mathcal{F}(g) \circ \mathcal{F}(f)$$

There is a "dual" type of functor, called **contravariant**, that reverses the arrows.

That is, they send $A \xrightarrow{\;f\;} B$ to $\mathcal{F}(B) \xrightarrow{\;\mathcal{F}(f)\;} \mathcal{F}(A)$

$$A \xrightarrow{\;f\;} B \xrightarrow{\;g\;} C \qquad\qquad \mathcal{F}(A) \xleftarrow{\;\mathcal{F}(f)\;} \mathcal{F}(B) \xleftarrow{\;\mathcal{F}(g)\;} \mathcal{F}(C)$$
$$g \circ f \qquad\qquad\qquad\qquad\qquad \mathcal{F}(g) \circ \mathcal{F}(f)$$

A contravariant functor from linear algebra

Let $V \in \mathrm{Ob}(\mathbf{Vect}_\mathbb{R})$ be an n-dimensional vector space.

The **dual space** $V^* \in \mathrm{Ob}(\mathbf{Vect}_\mathbb{R})$ consists of all *linear scalar functions* $\ell\colon V \to \mathbb{R}$.

Think of:

- elements in V as columns vectors,
- elements in V^* as row vectors.

$$\ell\colon V \longrightarrow \mathbb{R}, \qquad \ell(v) = \underbrace{\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}}_{\ell \in V^*} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{v \in V} = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n.$$

A linear map $A\colon V \to W$ can be represented by an $m \times n$ matrix, where $\dim(W) = m$.

Think of this as *left-multiplication by column vectors*, $Av = w$:

$$A\colon V \longrightarrow W, \qquad \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A \in \mathrm{Hom}(V,W)} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{v \in V} = \underbrace{\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}}_{w \in W}.$$

A contravariant functor from linear algebra

The *transpose* is a linear map $A^t \colon W^* \to V^*$.

Think of this as *right-multiplication by row vectors*, $w^t A^t = v^t$:

$$A^t \colon W^* \longrightarrow V^*, \qquad \underbrace{\begin{bmatrix} w_1 & w_2 & \cdots & w_m \end{bmatrix}}_{w^t \in W^*} \underbrace{\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{bmatrix}}_{A^t \in \mathsf{Hom}(W^*, V^*)} = \underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}}_{v^t \in V^*}$$

Formally, we have a contravariant functor:

$$\mathcal{F} \colon \mathsf{Ob}(\mathbf{Vect}_{\mathbb{F}}) \longrightarrow \mathsf{Ob}(\mathbf{Vect}_{\mathbb{F}}) \qquad \mathcal{F} \colon \mathsf{Hom}(\mathbf{Vect}_{\mathbb{F}}) \longrightarrow \mathsf{Hom}(\mathbf{Vect}_{\mathbb{F}})$$

$$V \longmapsto V^* \qquad\qquad v \overset{A}{\mapsto} w \quad \longmapsto \quad w^t \overset{A^*}{\mapsto} v^t$$

Notice how the arrow on the bottom of the following commutative diagram is reversed; this is contravariance.

$$
\begin{array}{ccc}
V & \xrightarrow{\quad A \quad} & W \\
{\scriptstyle \mathcal{F}} \downarrow & & \downarrow {\scriptstyle \mathcal{F}} \\
V^* & \xleftarrow{\quad A^* \quad} & W^*
\end{array}
\qquad
\begin{array}{l}
A \in \mathsf{Hom}(V, W) \\[1.5em]
A^* \in \mathsf{Hom}(W^*, V^*)
\end{array}
$$

Abelianization, as a functor from **Grp** to **Ab**

Consider the functor sending a group G to its abelianization $A \cong G/G' = G/[G, G]$:

$$\mathcal{F}: \text{Ob}(\mathbf{Grp}) \longrightarrow \text{Ob}(\mathbf{Ab}) \qquad \mathcal{F}: \text{Hom}(\mathbf{Grp}) \longrightarrow \text{Hom}(\mathbf{Ab})$$

$$G \longmapsto G/G' \qquad\qquad g \mapsto f(g) \longmapsto gG' \mapsto f(g)H'$$
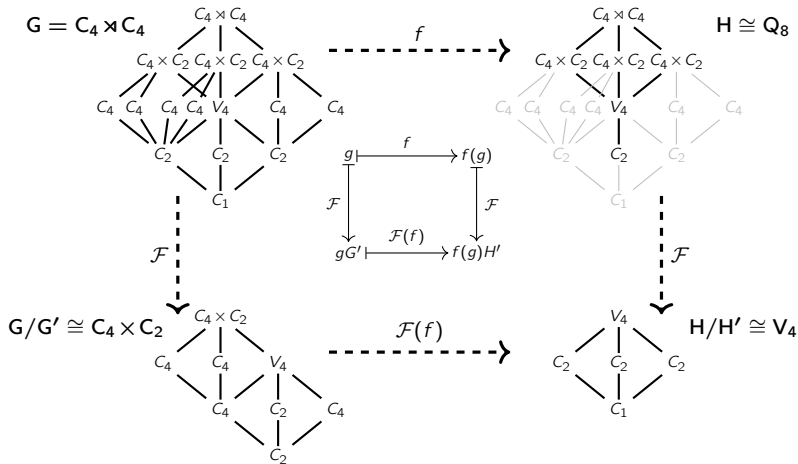
Initial and terminal objects

Definition

An object $I \in \mathrm{Ob}(\mathcal{C})$ is **initial** if for each $C_i \in \mathrm{Ob}(\mathcal{C})$, there is a unique $\pi_i \in \mathrm{Hom}_{\mathcal{C}}(I, C_i)$.

An object $T \in \mathrm{Ob}(\mathcal{C})$ is **terminal** if for each $C_i \in \mathrm{Ob}(\mathcal{C})$, there is a unique $\iota_i \in \mathrm{Hom}_{\mathcal{C}}(C_i, T)$.

An object that is initial and terminal is called a **zero object**.

Sometimes, initial objects are called *universal* or *coterminal*, and terminal objects are *final* or *couniversal*.

Category	Objects	Initial objects	Terminal objects	Zero objects
Set	sets	$\emptyset$	every $\{x\}$	none
Grp	groups	$\langle e \rangle$	$\langle e \rangle$	$\langle e \rangle$
Ab	abelian groups	$\langle 0 \rangle$	$\langle 0 \rangle$	$\langle 0 \rangle$
Rng	rings	$\{0\}$	$\{0\}$	$\{0\}$
Ring	rings w/ 1	$\mathbb{Z}$	$\{0\}$	none
Field	fields	none	none	none
Field$_p$	fields w/ char. $p > 0$	$\mathbb{Z}_p$	none	none
Vect$_{\mathbb{F}}$	$\mathbb{F}$-vector spaces	$\{0\}$	$\{0\}$	$\{0\}$
Top	topological spaces	$\emptyset$	every $\{x\}$	none

Initial and terminal objects

Proposition

Any two initial objects in a category $\mathcal{C}$ are equivalent.

Proof

Let I and J be initial.

Since I is initial, there is a unique morphism $f \in \text{Hom}_{\mathcal{C}}(I, J)$.

Since J is initial, there is a unique morphism $g \in \text{Hom}_{\mathcal{C}}(J, I)$.

The morphism $g \circ f$ is in $\text{Hom}_{\mathcal{C}}(I, I)$, as is Id_I (below, left).

$$I \underset{\text{Id}_I}{\overset{g \circ f}{\rightrightarrows}} I \qquad J \underset{\text{Id}_J}{\overset{f \circ g}{\rightrightarrows}} J$$

However, since I is initial, there must be a unique morphism in $\text{Hom}_{\mathcal{C}}(I, I)$, so $g \circ f = \text{Id}_I$.

Similarly, $f \circ g$ and Id_J are both in $\text{Hom}_{\mathcal{C}}(J, J)$ (above, right).

By uniqueness, $f \circ g = \text{Id}_J$, hence $I \cong J$. □

Uniquess of products

Suppose $\{B_i \mid i \in I\}$ in $\mathcal{C}$ has product P, with projections $\pi_i \colon P \to B_i$.

Define a new category $\mathcal{B}$:

- objects: families of maps $\{A \xrightarrow{f_i} B_i\}$
- morphisms: $A \xrightarrow{g} C$ that makes the following diagram commute.

$$g \in \mathsf{Hom}_{\mathcal{B}}\left(A \xrightarrow{f_i} B_i,\ C \xrightarrow{h_i} B_i\right)$$

$$
\begin{array}{ccc}
A & \xrightarrow{\ f_i\ } & B_i \\
& \searrow{\scriptstyle g} \quad \nearrow{\scriptstyle h_i} & \\
& C &
\end{array}
$$

A terminal object in $\mathcal{B}$ is a family $\{P \xrightarrow{\pi_i} B_i\}$ such that for any $\{A \xrightarrow{f_i} B_i\}$, there exists a unique $g \in \mathsf{Hom}_{\mathcal{C}}(A, P)$ that makes the diagram (left) commute:

$$
\begin{array}{ccc}
A & \xrightarrow{\ f_i\ } & B_i \\
{\scriptstyle \exists! g}\searrow & & \nearrow{\scriptstyle \pi_i} \\
& P &
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & \xrightarrow{\ f_i\ } & B_i \\
{\scriptstyle \exists! g}\searrow & & \nearrow{\scriptstyle \pi_i} \\
& \prod_{i\in I} B_i &
\end{array}
$$

That is, the terminal object is the product! Thus, products are unique up to equivalence.

Uniquess of coproducts and zero morphisms

We can construct an analogous category where the initial object is the coproduct.

$$A_i \xrightarrow{\ f_i\ } B \qquad\qquad A \xrightarrow{\ f_i\ } B_i$$

"initial" $g_i \searrow S \nearrow \exists! h$ $\qquad\qquad$ $\exists! g \searrow P \nearrow h_i$ "terminal"
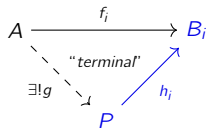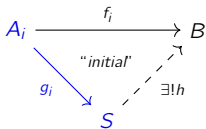
$$h \in \mathsf{Hom}_{\mathcal{A}}\left(A_i \xrightarrow{\ g_i\ } S, A_i \xrightarrow{\ f_i\ } B\right) \qquad g \in \mathsf{Hom}_{\mathcal{B}}\left(A \xrightarrow{\ f_i\ } B_i, P \xrightarrow{\ h_i\ } B_i\right)$$

Though each $\pi_i \in \mathsf{Hom}_{\mathcal{C}}(P, B_i)$ need not be epic, there are conditions that guarantee this.

Definition
Let $\mathcal{C}$ be a category with a zero object, $\mathbf{0} \in \mathsf{Ob}(\mathcal{C})$. The **zero morphism** $0_{AB} \in \mathsf{Hom}_{\mathcal{C}}(A, B)$ is the composition of the unique maps $A \to \mathbf{0} \to B$.

$$A \xrightarrow{\ 0_{AB}\ } B$$

$f \downarrow \quad \searrow^{0_{AC}} \quad \downarrow g$

$$B \xrightarrow{\ 0_{B\mathbf{0}}\ } C$$
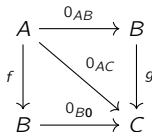
Zero morphisms

Proposition

If $0 \in \mathrm{Ob}(\mathcal{C})$, the projection morphisms $\pi_i \in \mathrm{Hom}_{\mathcal{C}}(P, B_i)$ of a product are epimorphisms.

Proof

Fix $\alpha \in I$, and define the family of maps $\left\{ f_i \in \mathrm{Hom}_{\mathcal{C}}(B_\alpha, B_i) \mid i \in I \right\}$ as

$$f_i \colon B_\alpha \longrightarrow B_i, \qquad f_i = \begin{cases} \mathsf{Id}_{B_\alpha}, & i = \alpha \\ 0_{B_i \mathbf{0}}, & i \neq \alpha. \end{cases}$$

By the universal property of products, for each $i \in I$, we have:

$$
\begin{array}{ccc}
B_\alpha & \xrightarrow{\ f_i = \mathsf{Id}_{B_\alpha}\ } & B_i \\
& \exists! g_i = g_\alpha \searrow \quad i = \alpha \quad \nearrow \pi_i = \pi_\alpha & \\
& P &
\end{array}
\qquad\qquad
\begin{array}{ccc}
B_\alpha & \xrightarrow{\ f_i = 0_{B_i \mathbf{0}}\ } & B_i \\
& \exists! g_i \searrow \quad i \neq \alpha \quad \nearrow \pi_i & \\
& P &
\end{array}
$$

To show π_α is epic, we need to verify left-cancellation.

Consider $f, g \in \mathrm{Hom}_{\mathcal{C}}(B_\alpha, C)$ such that $f \circ \pi_\alpha = g \circ \pi_\alpha$.

Zero morphisms

Proposition

If $0 \in \text{Ob}(\mathcal{C})$, the projections $\pi_i \in \text{Hom}_{\mathcal{C}}(P, B_i)$ from a product are epimorphisms.

Proof

It suffices to show that $f = g$.

$$B_\alpha \xrightarrow{\ f_\alpha \,=\, \text{Id}_{B_\alpha}\ } B_\alpha \underset{g}{\overset{f}{\rightrightarrows}} C$$

with g_α from B_α to P and π_α from P to B_α.

By the commuativity of the diagram, we have

$$f = f \circ \text{Id}_{B_\alpha} = f \circ (\pi_\alpha \circ g_\alpha) = (f \circ \pi_\alpha) \circ g_\alpha = (g \circ \pi_\alpha) \circ g_\alpha = g \circ (\pi_\alpha \circ g_\alpha) = g \circ \text{Id}_{B_\alpha} = g,$$

whence π_α is an epimorphism. $\qquad\square$

Proposition (HW)

If $0 \in \text{Ob}(\mathcal{C})$, the inclusions $\iota_i \in \text{Hom}_{\mathcal{C}}(B_i, S)$ into a coproduct are monomorphisms.

Free groups

Throughtout, let S be a nonempty set.

Definition

The **free group on S** is
$$F = F_S := \langle S \mid \ \rangle.$$
That is, F_S is generated by S, subject to no relations.

We can think of the free groups as groups where:

- elements are words in $T = S \sqcup S^{-1}$, where $S^{-1} := \{s^{-1} \mid s \in S\}$.

- the binary operation is concatenation.

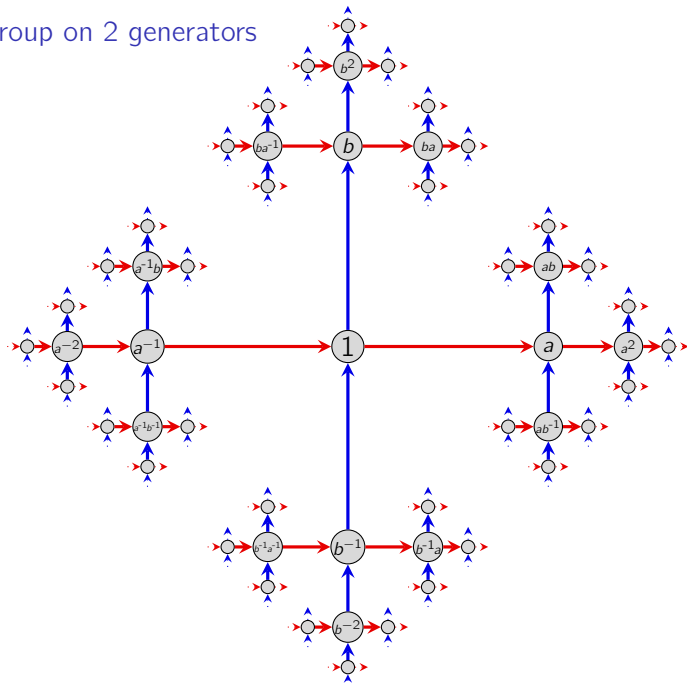The only way to modify words are by substitutions of form $ss^{-1} = 1$ and $s^{-1}s = 1$.

If $|S| = |T|$, then $F_S \cong F_T$.

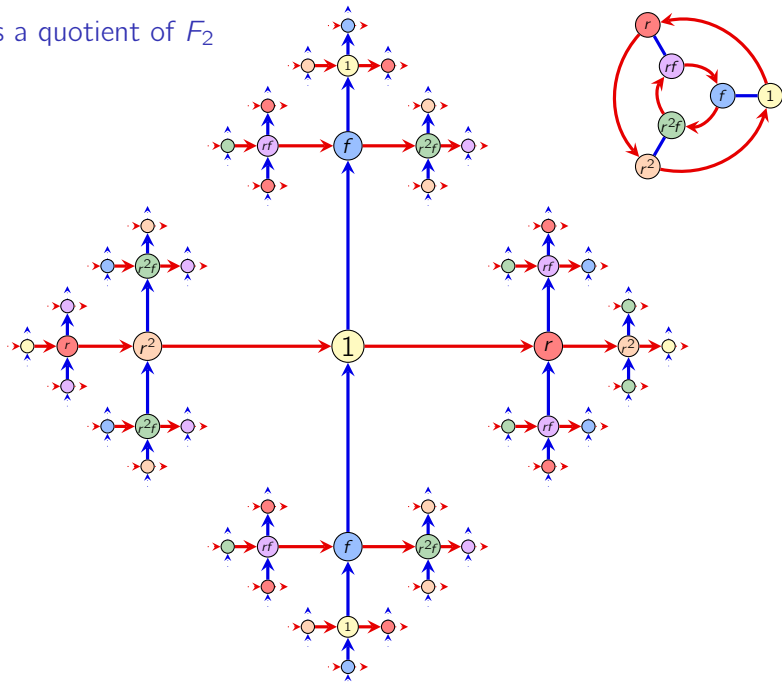If $|S| = n < \infty$, then $F_n := F_S$ is *free group on n generators*, or the *free group of rank n*.

We'll soon see how every group is a quotient of a free group.

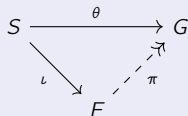This can be formalized via a couniversal property.

The free group on 2 generators

Free groups

Definition

A group F is **free** on $S \neq \emptyset$ if there is a function $\iota \colon S \to F$ such that for any other $\theta \colon S \to G$, there exists a unique homomorphism $\pi \colon F \to G$ such that $\theta = \pi \circ \iota$.

$$S \xrightarrow{\;\;\theta\;\;} G$$

with maps $\iota \colon S \to F$ and $\pi \colon F \to G$ forming a commutative triangle.

Proposition

If a free group exists on $S \neq \emptyset$, it is unique up to isomorphism, and $\iota \colon S \to F$ is injective.

Proof

We've seen uniqueness. Suppose ι is not 1-to-1; take $a \neq b$ in S for which $\iota(a) = \iota(b)$.

Consider the map $\theta \colon S \longrightarrow \mathbb{Z}$, $\qquad \theta(s) = \begin{cases} 1 & s = a \\ 2 & s = b \\ 0 & s \notin \{a, b\}. \end{cases}$

This forces $1 = \theta(a) = \pi(\iota(a)) = \pi(\iota(b)) = \theta(b) = 2$, a contradiction. $\qquad\square$
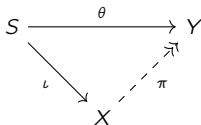
Free semigroups

Definition

A **semigroup** is a set $X \neq \emptyset$ with associative binary operation.

A **homomorphism** is a function $f \colon X \to Y$ with $f(x_1 x_2) = f(x_1) f(x_2)$ for all $x_1, x_2 \in X$.

Let **Sgp** denote the category of semigroups.

Free semigroups exists, are unique up to isomorphism, the map $\iota \colon S \to F$ is injective.

$$
\begin{array}{ccc}
S & \xrightarrow{\quad \theta \quad} & Y \\
& \searrow_{\iota} \quad \nearrow_{\pi} & \\
& X &
\end{array}
$$

The free semigroup on $S = \{s\}$ is isomorphic to $\mathbb{N} = \{1, 2, \dots\}$ under addition.

Free semigroups

Proposition

If $S \neq \emptyset$, then there is a free semigroup over S.

Proof

Let X be the set of nonempty words over S, under concatenation:

$$X = S \cup (S \times S) \cup (S \times S \times S) \cup \cdots, \qquad (a_1, \ldots, a_n) * (b_1, \ldots, b_m) = (a_1, \ldots, a_n, b_1, \ldots, b_m).$$
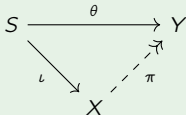
We'll show this is free over S, with inclusion map

$$\iota \colon S \longrightarrow X, \qquad \iota(s) = s.$$

Given a function $\theta \colon S \to Y$ to another semigroup, define

$$\pi \colon X \longrightarrow Y, \qquad \pi \colon (a_1, \ldots, a_n) \longmapsto \theta(a_1) \cdots \theta(a_n).$$

Exercise. Check that π is a semigroup homomorphism, and $\pi \circ \iota = \theta$.

Free semigroups

Proposition

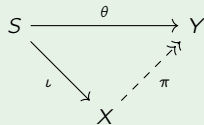If $S \neq \emptyset$, then there is a free semigroup over S.

Proof (contin.)

Given $\theta \colon S \to Y$, the function

$$\pi \colon X \longrightarrow Y, \qquad \pi \colon (a_1, \ldots, a_n) \longmapsto \theta(a_1) \cdots \theta(a_n).$$

satisfies $\pi \circ \iota = \theta$.

Uniqueness: Suppose $\sigma \colon X \to Y$ also satisfies $\sigma \circ \iota = \theta$. Then

$$\begin{aligned}
\sigma\big((a_1, \ldots, a_n)\big) &= \sigma(\iota(a_1) \cdots \iota(a_n)) \\
&= \sigma(\iota(a_1)) \cdots \sigma(\iota(a_n)) \\
&= \theta(a_1) \cdots \theta(a_n) \\
&= \pi(\iota(a_1)) \cdots \pi(\iota(a_n)) \\
&= \pi(\iota(a_1) \cdots \iota(a_n)) \\
&= \pi\big((a_1, \ldots, a_n)\big).
\end{aligned}$$

Therefore, X satisfies the co-universal property of free semigroups. $\qquad\qquad\square$
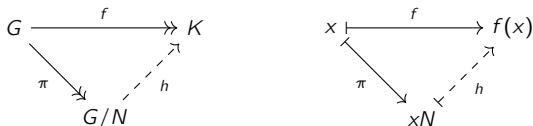
Quotient semigroups

Since semigroups lack an inverse, we don't have kernels, or isomorphism theorems.

But there is a co-universal property of quotient maps.

A group homomorphism $f\colon G \to K$ partitions G into cosets of $\mathsf{Ker}(f)$.

If this is coarser than the partition of G into cosets of $N = \mathsf{Ker}(\pi)$, then f factors through π:

$$
\begin{array}{ccc}
G & \xrightarrow{\ \ f\ \ } & K \\
\pi \searrow & & \nearrow h \\
& G/N &
\end{array}
\qquad\qquad
\begin{array}{ccc}
x & \xmapsto{\ \ f\ \ } & f(x) \\
\pi \searrow & & \nearrow h \\
& xN &
\end{array}
$$

A relation R on a semigroup Y is **well-defined** with respect to $\cdot$ if

$$x R y \quad \text{and} \quad z R w \quad \Longrightarrow \quad (x \cdot z) R (y \cdot w).$$

Let xR be the equivalence class containing x, and call

$$\pi\colon Y \longrightarrow Y/R, \qquad \pi\colon y \longmapsto yR$$
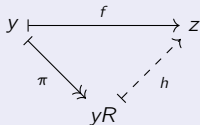
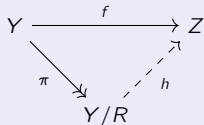the **canonical quotient map**.

The **quotient semigroup** of Y is Y/R, with $xR \cdot yR := xyR$.

Co-universal property of quotient semigroups

Proposition

The quotient semigroup Y/R satisfies the following co-universal property:

If $f\colon Y \to Z$ is a semigroup homomorphism such that xRy implies $f(x) = f(y)$, then $\exists! h\colon Y/R \to Z$ such that $f = h \circ \pi$.

Proof

Existence follows from the definition: $h(yR) = h(\pi(y)) = f(y)$, with well-definedness automatic from $xRy \Rightarrow f(x) = f(y)$.

Uniqueness comes from the cancellation laws, because π is surjective. $\qquad\square$

Construction of a free group over S

Given $S \neq \emptyset$, construct a disjoint set S' of "formal inverses":

$$S' = \{s' \mid s \in S\}, \qquad T = S \cup S'.$$

The bijection $s \mapsto s'$ and inverse $s' \mapsto s'' := s$ define a bijection $T \to T$, where $t \mapsto t'$.

Let X be the free semigroup on $T \subseteq X$ (under natural inclusion).

Call a homomorphism $\phi \colon X \to G$ **proper** if $\phi(s') = \phi(s)^{-1}$ for all $s \in S$.

If ϕ is proper, then $\phi(t') = \phi(t)^{-1}$ for all $t \in T$.

The only "relation" in a free group: $ss^{-1} = s^{-1}s = 1$ for all $s \in S \subseteq F$.

We'll construct this from the free semigroup by forcing $ss't = t$, for all $t \in T \subseteq X$.

If ϕ is proper, then

$$\phi(ss't) = \phi(s)\phi(s')\phi(t) = \phi(s)\phi(s)^{-1}\phi(t) = \phi(t).$$

Define an equivalence relation on X where

$$xRy \quad \text{iff} \quad \phi(x) = \phi(y) \text{ for every proper } \phi \colon X \to G.$$

Exercise: this is well-defined, and so X/R is a semigroup. .

Construction of a free group over S

We just showed that X/R is a semigroup. Now we'll show it's a group.

We'll write $\bar{x}$ (not xR), so $\bar{x}\bar{y} = \overline{xy}$, and $\overline{x'} = \bar{x}^{-1}$.

Let $\pi\colon X \twoheadrightarrow X/R$ be the canonical quotient.

Identity. Choose any $s \in S$ and $x \in X$; we claim that $\overline{ss'} = 1$.

If $\phi\colon X \to G$ is proper, then $\phi(ss'x) = \phi(x)$, which means that $xRss'x$ in X, thus

$$\bar{x} = \overline{ss'x} = \overline{ss'} \cdot \bar{x}, \qquad \text{and} \qquad \bar{x} = \overline{xss'} = \bar{x} \cdot \overline{ss'}.$$

Thus, $\overline{ss'}$ is the identity. $\checkmark$

Inverses. Let $x = t_1 \cdots t_k \in X$.

We'll show that the inverse of $\bar{x}$ is $\bar{y}$, where $y = t'_k \cdots t'_1$.

If ϕ is proper, then

$$\begin{aligned}
\phi(xy) &= \phi(t_1 \cdots t_k t'_k \cdots t'_1) \\
&= \phi(t_1) \cdots \phi(t_k)\phi(t'_k) \cdots \phi(t'_1) \\
&= \phi(t_1) \cdots \phi(t_k)\phi(t_k)^{-1} \cdots \phi(t_1)^{-1} \\
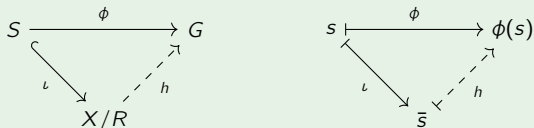&= 1_G = \phi(ss') \quad \text{for any } s \in S.
\end{aligned}$$
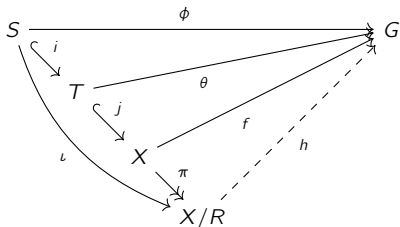
$\checkmark$

Thus X/R is a group. $\qquad\square$

Showing that our free semigroup quotient X/R is free

Goal

Given $\iota\colon S \to X/R$ defined by $\iota(s) = \bar{s}$, show that for any map $\phi\colon S \to G$, there is a unique homomorphism $h\colon X/R \to G$ such that $\phi = h \circ \iota$

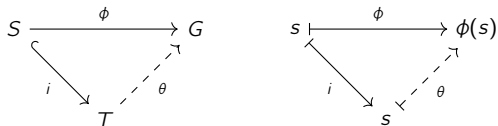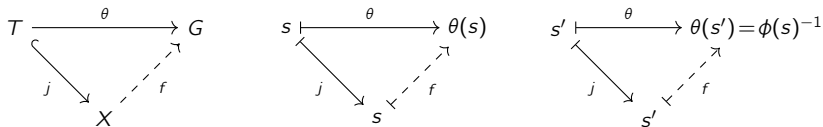We'll build up this diagram in "pieces", culminating with the following:

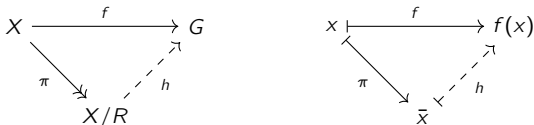## Showing that our free semigroup quotient X/R is free

Extend $\phi\colon S \to G$ to a map $\theta\colon T \longrightarrow G$ by setting $\theta(s') = \phi(s)^{-1}$.

Applying the co-universal property of free semigroups to $\theta\colon T \to G$ gives the following:
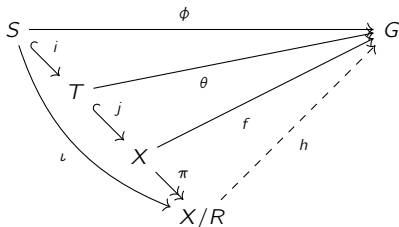

Since the homomorphism f is proper, the co-universal property of quotient semigroups gives:

Showing that our free semigroup quotient X/R is free

We know $\exists! h \colon X/R \to G$ such that $f = h \circ \pi$, but not necessarily $\phi = h \circ \iota$.

Suppose $\exists g \colon X/R \to G$ such that $\phi = g \circ \iota$. (Need $h = g$.)

We have $h \circ \pi \circ j \circ i = g \circ \pi \circ j \circ i$, and we claim that $h \circ \pi \circ j = g \circ \pi \circ j$.

It is clear that $h(\pi(j(s))) = f(\pi(j(s))$ for all $s \in S$. By construction,

$$h(\pi(j(s'))) = h(\overline{s'}) = h(\overline{s}^{-1}) = h(s)^{-1} = g(s)^{-1} = g(\overline{s}^{-1}) = g(\overline{s'}) = g(\pi(j(s'))).$$

Therefore, $\theta = h \circ \pi \circ j = g \circ \pi \circ j$.

By the co-universal property of free semigroups, $\exists! f \colon X \to G$ such that $\theta = f \circ j$.

But both $h \circ \pi$ and $g \circ \pi$ satisfy this, and so $f = h \circ \pi = g \circ \pi \;\Rightarrow\; h = g$ $\qquad\square$

Properties of free groups

Proposition

Suppose $S, U \neq \emptyset$. Then $F_S \cong F_U$ if and only if $|S| = |U|$.

Proof

"$\Rightarrow$" **Case 1**: $|S| < \infty$.

Each nonempty $R \subseteq S$ defines an index-2 subgroup, the kernel of

$$f_R \colon F_S \longrightarrow \mathbb{Z}_2, \qquad f_R(s) = \begin{cases} 0 & s \in R \\ 1 & s \notin R \end{cases}$$

Since F_U has the same number of index-2 subgroups, $2^{|S|} - 1 = 2^{|U|} - 1 \Rightarrow |S| = |U|$.

Case 2: $|S| = \infty$.

Let $T = S \cup S^{-1}$. Then $|F_S| = |S|$ because.

$$|F_S| \leq 1 + |T| + |T \times T| + |T \times T \times T| + \cdots = \aleph_0 |T| = |S|.$$
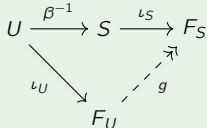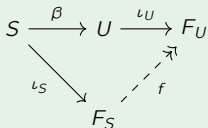
Reversing roles gives $|F_U| = |U| = |S| = |F_S|$. $\checkmark$

Properties of free groups

Proposition
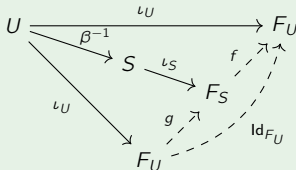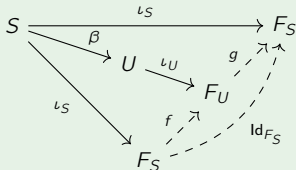
Suppose $S, U \neq \emptyset$. Then $F_S \cong F_U$ if and only if $|S| = |U|$.

Proof

"$\Leftarrow$" Fix a bijection $\beta \colon S \to U$ and use the co-universal property to get

$$S \xrightarrow{\ \beta\ } U \xrightarrow{\ \iota_U\ } F_U \qquad\qquad U \xrightarrow{\ \beta^{-1}\ } S \xrightarrow{\ \iota_S\ } F_S$$

$$\iota_S \searrow \quad \nearrow f \qquad\qquad\qquad \iota_U \searrow \quad \nearrow g$$

$$F_S \qquad\qquad\qquad\qquad\qquad F_U$$

We can "stack" these diagrams, two ways, to get:

By uniqueness, $g \circ f = \mathsf{Id}_{F_S}$ and $f \circ g = \mathsf{Id}_{F_U}$, so f and g are inverse isomorphisms. $\square$
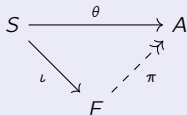
Free objects

A functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ is **faithful** if $\mathcal{F}\colon \mathsf{Hom}(\mathcal{C}) \to \mathsf{Hom}(\mathcal{D})$ is injective.

A concrete category is a category $\mathcal{C}$ with a faithful functor $\mathcal{F}\colon \mathcal{C} \to \mathsf{Set}$.
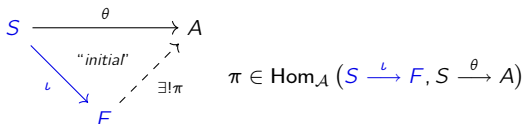
Definition

Let $\mathcal{C}$ be concrete, $F \in \mathsf{Ob}(\mathcal{C})$, and $\iota\colon S \to F$ a map of sets, where $S \neq \emptyset$.

Then F is free on S if for any $A \in \mathsf{Ob}(\mathcal{C})$ and $\theta\colon S \to A$, there is a unique $\pi \in \mathsf{Hom}_\mathcal{C}(F, A)$ such that $\iota \circ \pi = \theta$.

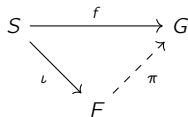Like we did with products, we can construct a category where free objects are initial:

$$\pi \in \mathsf{Hom}_\mathcal{A}\left(S \xrightarrow{\ \iota\ } F, S \xrightarrow{\ \theta\ } A\right)$$

Free objects

Let **Nil** be the category of nilpotent groups.

Suppose $\iota \colon S \to F$ is a free object in **Nil**.

This means that every other nilpotent group G generated by S is a quotient of F:

> "if G is nilpotent with set map $f \colon S \to G$, then there exists a unique $\pi \colon F \to G$ such that $f = \pi \circ \iota$."

$$
\begin{array}{ccc}
S & \xrightarrow{\quad f \quad} & G \\
& \searrow{\iota} \quad \nearrow{\pi} & \\
& F &
\end{array}
$$

Suppose F has nilpotency class n. Then every quotient has nilpotency class $\leq n$. (Why?)

Thus, if $G = \langle S \rangle$ has nilpotency class $n+1$, then $\nexists\, \pi \colon F \twoheadrightarrow G$.

Free objects

Let $\mathbf{Nil}_{\leq n}$ be the category of nilpotent groups of class $\leq n$.

If G is a nilpotent group of class $\leq n$, then $L_n(G) = \langle 1 \rangle$.

$$
\begin{aligned}
L_1(G) &= [G, L_0] &&= [G, G] &&= \langle [g_1, g_0] \mid g_i \in G \rangle \\
L_2(G) &= [G, L_1] &&= [G, [G, G]] &&= \langle [g_2, [g_1, g_0]] \mid g_i \in G \rangle \\
L_3(G) &= [G, L_2] &&= [G, [G, [G, G]]] &&= \langle [g_3, [g_2, [g_1, g_0]]] \mid g_i \in G \rangle \\
&\ \ \vdots &&\ \ \vdots &&\ \ \vdots \\
L_n(G) &= [G, L_{k-1}] &&= [G, [G, \ldots [G, G]]] &&= \langle [g_n, [g_{n-1}, \ldots [g_1, g_0]]] \mid g_i \in G \rangle
\end{aligned}
$$

Proposition

Let F be free on a set S. Then $F/L_n(F)$ is free in $\mathbf{Nil}_{\leq n}$.

"if G is nilpotent of class $\leq n$ and $f \colon S \to G$, then there exists a unique $\pi \colon F_n/L_n(F_n) \to G$ such that $f = \pi \circ \iota$."
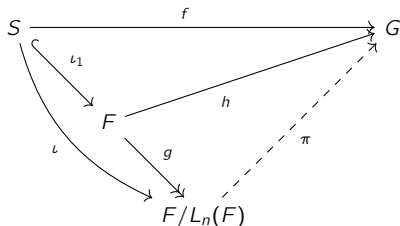
Free objects

Proposition

Let F be free on S. Then $F/L_n(F)$ is free in $\mathbf{Nil}_{\leq n}$.

The existence of $h\colon F \to G$ is because F is free on S.

Since G has nilpotent class $\leq n$, we have $\mathrm{Ker}(g) = L_n(F) \leq \mathrm{Ker}(h)$.

Now, π is guaranteed by the co-universal property of quotient maps.

Exercise: Verify that π is the unique map satsifying $f = \pi \circ \iota$.

Direct sums and bases

The direct sum of a family $\{A_i \mid i \in I\}$ of groups is

$$\bigoplus_{i \in I} A_i = \left\{ (a_i)_{i \in I} \in \prod_{i \in I} A_i \ \text{ with finite support} \right\}.$$

If all are abelian, let $\mathbf{e}_j := (a_i)_{i \in I}$ with $a_j = \delta_{ij}$. Every $x \in \bigoplus A_i$ can be written as

$$x = \sum_{i=1}^{n} a_i \mathbf{e}_i, \qquad a_i \in \mathbb{Z}, \quad n \in \mathbb{N}.$$

If A is abelian, the subgroup generated by $X \subseteq S$ are the finite linear combinations:

$$\langle X \rangle = \left\{ a_1 x_1 + \cdots + a_n x_n \mid a_i \in \mathbb{Z}, \ x_i \in X \right\}.$$

A basis of A is a subset $X \subseteq A$ for which:

1. $A = \langle X \rangle$.

2. Given distinct $x_1, \ldots, x_n \in X$,

$$a_1 x_1 + \cdots + a_n x_n = 0 \qquad \Longrightarrow \qquad a_i = 0 \ \text{ for all } i = 1, \ldots, n.$$

Direct sums and bases

Assuming the axiom of choice, in a vector space, Every generating set contains a basis.

This fails for abelian groups; e.g., $\mathbb{Z} = \langle 2, 3 \rangle$.

Every vector space has a basis, and every $v \neq 0$ is contained in one.

If an abelian group A has an element x of finite order, no basis can contain it.

Proposition

Let A be an abelian group with basis X. Then every $a \in A$ can be written as a unique (finite) linear combination of elements from X.

Proof

The following defines a homomorphism

$$f : \bigoplus_{i \in I} \mathbb{Z} \longrightarrow A, \qquad f : \sum_{j=1}^{n} a_j \mathbf{e}_j \longmapsto \sum_{j=1}^{n} a_j x_j.$$

It is surjective by Property (1) of a basis, and has trivial kernel by Property (2).

Each way to write x as a linear combination of the basis elements corresponds to an f-preimage of x.

Uniqueness follows because f is bijective. $\qquad\square$
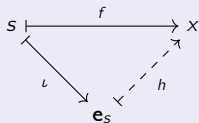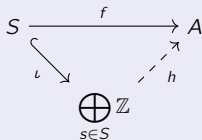
Free abelian groups

Definition

The **free abelian group** on $S \neq \emptyset$ is $\quad F = \bigoplus_{s \in S} \mathbb{Z}$.

Theorem

Let $S \neq \emptyset$. The group $\bigoplus_{s \in S} \mathbb{Z}$ with $\iota(s) = \mathbf{e}_s$ is a free object for S in **Ab**.

That is, given any $f \colon S \to A$ there exists a unique $h \colon \bigoplus_s \mathbb{Z} \to A$ such that $f = h \circ \iota$.

Proof (sketch)

Existence and uniqueness of the desired function h is constructive:

$$h\Big(\sum_{i=1}^n a_{s_i} \mathbf{e}_{s_i} \Big) = \Big(\sum_{i=1}^n a_{s_i} h(\mathbf{e}_{s_i}) \Big) = \Big(\sum_{i=1}^n a_{s_i} h(\iota(s_i)) \Big) = \sum_{i=1}^n a_{s_i} f(s_i). \qquad \square$$

Group presentations, formalized

Definition

For any subset $R \subseteq F_S$, the group $G = \langle S \mid R \rangle$ is the quotient F_S / N where

$$N := \bigcap_{R \leq N_\alpha \trianglelefteq F_S} N_\alpha.$$

Elements in R are called relators.
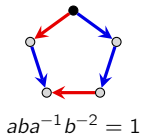
Big idea

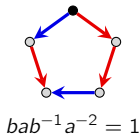The group $\langle S \mid R \rangle$ is the quotient of F_S by the *smallest normal subgroup containing R*.

Exercise: show that

$$G = \langle a, b \mid ab = b^2 a, \; ba = a^2 b \rangle = \langle 1 \rangle.$$

In terms of Cayley graphs and motfis, this means that

$aba^{-1}b^{-2} = 1$ and $bab^{-1}a^{-2} = 1$ $\implies$ $G = \langle a, b, \mid a = b = 1 \rangle = \langle 1 \rangle$

Group presentations, formalized

Given $G_1 = \langle S \mid R_1 \rangle$, define $G_2 = \langle S \mid R_2 \rangle$ by adding relations: $R_1 \subseteq R_2$.

We have two quotient maps,

$$\pi_1 \colon F_S \longrightarrow F_S/N_1 \cong G_1, \qquad\qquad \pi_2 \colon F_S \longrightarrow F_S/N_2 \cong G_2,$$

Since $N_1 = \mathrm{Ker}(\pi_1) \le \mathrm{Ker}(\pi_2) = N_2$, the co-universal property of quotients gives us:

Now, suppose $G_1 = \langle S_1 \mid R \rangle$ and $G_2 = \langle S_2 \mid R \rangle$ with $S_1 \supseteq S_2$.

Defining $R' = S_1 \setminus S_2$, we have

$$G_1 = \langle S_1 \mid R \rangle, \qquad G_2 = \langle S_1 \mid R \cup R' \rangle,$$

and hence a quotient $G_1 \twoheadrightarrow G_2$.

Proposition

Given $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$ for which $S_1 \supseteq S_2$ and $R_1 \subseteq R_2$, there is a quotient $G_1 \twoheadrightarrow G_2$. $\qquad\square$

Group presentations, formalized

In many cases, two generating sets that we wish to compare are not subsets of each other.

For example, if $S_1 = \{a, b, c\}$ and $S_2 = \{r, f\}$, then $S_1 \not\supseteq S_2$.

However, there is $\theta\colon S_1 \twoheadrightarrow S_2$ that can be thought of as a "relabeling."

Saying that "*every relation is G_1 is a relation in G_2*" means that every $\theta(r_1)$ is a relator.

We say that such a map θ *respects relations*, because it extends to a map $\theta\colon R_1 \to R_2$.
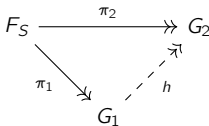
Proposition

Suppose $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$ and the following holds:

1. there exists $\theta\colon S_1 \twoheadrightarrow S_2$ extending to $\theta\colon R_1 \to R_2$,
2. $r_2 := \theta(r_1) = 1$ for all $r \in R_1$.

Then there is a quotient $h\colon G_1 \twoheadrightarrow G_2$. $\qquad\qquad\square$
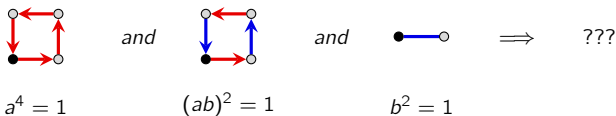
Group presentations, formalized

Consider the "mystery group" $M = \langle a, b \mid a^4 = b^2 = 1, \ (ab)^2 = 1 \rangle$.

Visually, we are asking what the largest Cayley graph is given several motifs:

$a^4 = 1$ and $(ab)^2 = 1$ and $b^2 = 1$ $\implies$???

Elements in M can be written as $a^i b^j$, for $i = \{0, 1, 2, 3\}$ and $j = \{0, 1\}$. Thus, $|M| \leq 8$.

We'll show $|M|$ is a multiple of 8, by constructing a homomorphism

$$\theta \colon M \longrightarrow \left\langle \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle \cong D_4, \qquad \theta(a) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \theta(b) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This respects relations because

$$\left(\theta(a)\right)^4 = \left(\theta(b)\right)^2 = \left(\theta(a)\theta(b)\right)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, there is a quotient $g \colon M \twoheadrightarrow D_4$, and so $M \cong D_4$.

Group presentations, formalized

Every group $G = \langle a, b \rangle$ satisfying $a^4 = 1$, $b^2 = 1$, and $(ab)^2 = 1$ is a quotient of D_4.
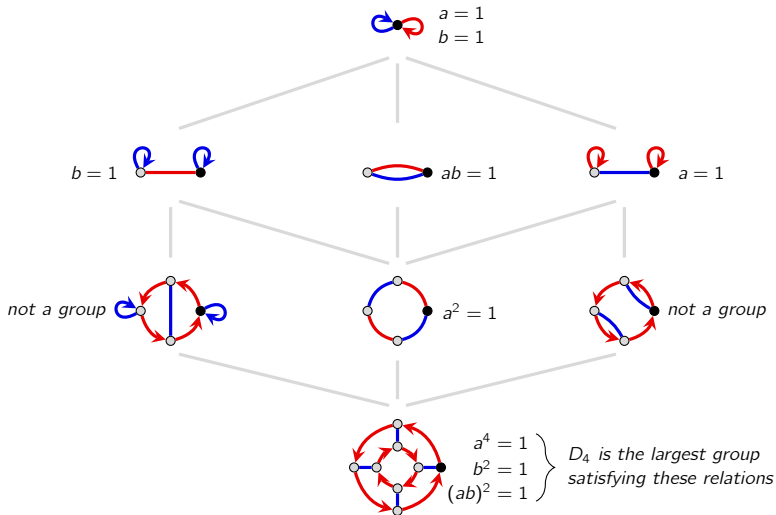
Group presentations, formalized

Overview of the strategy

Given a "mystery" $M = \langle S_1 \mid R_1 \rangle$ that we suspect is a "familiar" $F = \langle S_2 \mid R_2 \rangle$:

1. Using the relations, show that $|M| \leq |F|$.
2. Identify generators of F that satisfy the relations in M, via a "relabling map" $\theta \colon S_1 \twoheadrightarrow S_2$ that extends to $\theta \colon R_1 \rightarrow R_2$.

Together, $|M| \leq |F|$ and $M \twoheadrightarrow F$ forces $M \cong F$.

Consider the group $M = \langle a, b, c \mid a^4 = c^2 = 1, \; a^2 = b^2, \; ab = ba, \; ac = ca, \; a^2 b = cbc \rangle$.

$a^4 = 1$ $c^2 = 1$ $a^2 = b^2$ $ab = ba$ $ac = ca$ $a^2 b = cbc$

Homework: Establish $|M| \leq 16$ by showing that every word in M can be written

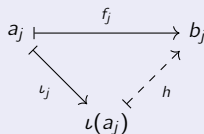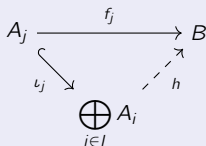$$a^i b^j c^k, \qquad i \in \{0, 1, 2, 3\}, \quad j \in \{0, 1\}, \quad k \in \{0, 1\},$$

Then, find a "familiar group" F of order 16 whose generator satisfies these relations.

That will define a quotient $\pi \colon M \twoheadrightarrow F$, and hence $|M| \geq |F| = 16$.

Free products

Proposition

The coproduct of $\{A_i \mid i \in I\}$ in **Ab** is the direct sum, $S = \bigoplus_i A_i$:

Proof

Let C be the coproduct of the factors, with $\iota_j \colon A_j \hookrightarrow C$.

Consider the group $B \leq C$ generated by the images of all individual factors,

$$B = \langle \iota_j(A_j) \mid j \in I \rangle, \qquad \text{and let } g \colon B \hookrightarrow C.$$

Each $b \in B$ can be written as

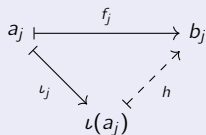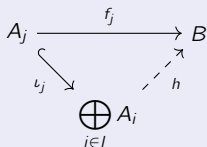$$b = \sum_{j=1}^{k} \iota(a_{i_j}), \qquad a_{i_j} \in A_{i_j},$$

and so $B \cong S$. Let $f_j \colon A_j \hookrightarrow B$ be the natural inclusion map.
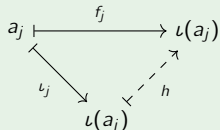
Free products

Proposition

The coproduct of $\{A_i \mid i \in I\}$ in **Ab** is the direct sum, $S = \bigoplus_i A_i$:

$$A_j \xrightarrow{\ f_j\ } B \qquad\qquad \bigoplus_{i \in I} A_i$$

$$a_j \longmapsto b_j \qquad\qquad \iota(a_j)$$

with maps ι_j, h.

Proof (cont.)

By the co-universal property of coproducts, we have:

$$A_j \xhookrightarrow{\ f_j\ } B \qquad\qquad C \cong S$$

$$a_j \longmapsto \iota(a_j) \qquad\qquad \iota(a_j)$$

with maps ι_j, h.

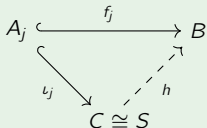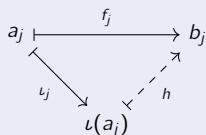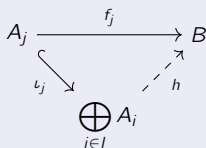It is clear that $h \circ g = \mathrm{Id}_B$. It suffices to show that $g \circ h = \mathrm{Id}_C$.

Free products
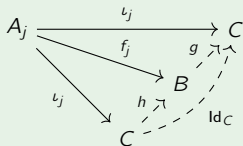
Proposition

The coproduct of $\{A_i \mid i \in I\}$ in **Ab** is the direct sum, $S = \bigoplus_i A_i$:

Proof (cont.)

Since $\iota_j = g \circ f_j$ and $f_j = h \circ \iota_j$, the "small triangles" in the following diagram commute:

It follows that $\iota_j = g \circ h \circ \iota_j$, but we also have $\iota_j = \mathsf{Id}_C \circ \iota_j$.

By uniqueness from the co-universal property, $g \circ h = \mathsf{Id}_C$. $\qquad\qquad \square$

Free products

The coproduct of two groups A and B in **Grp** is a construction called the **free product**.

Given groups $A = \langle S_1 \mid R_1 \rangle$ and $B = \langle S_2 \mid R_2 \rangle$, their *free product* is

$$A * B := \langle S_1 \sqcup S_2 \mid R_1 \sqcup R_2 \rangle.$$

If $A = \langle a \mid \, \rangle = C_\infty \cong \mathbb{Z}$ and $B = \langle b \mid \, \rangle \cong C_\infty$, then $A * B$ is the free group $F_2 = \langle a, b \mid \, \rangle$.

If A and B are nontrivial, their free product is infinite, because

$$a, \quad ab, \quad aba, \quad abab, \quad ababa, \quad ababab, \ldots$$

are all distinct, assuming $a, b \neq 1$.

The free product of the groups $A = \langle a \mid a^2 = 1 \rangle \cong C_2$ and $B = \langle b \mid b^2 = 1 \rangle \cong C_2$ is

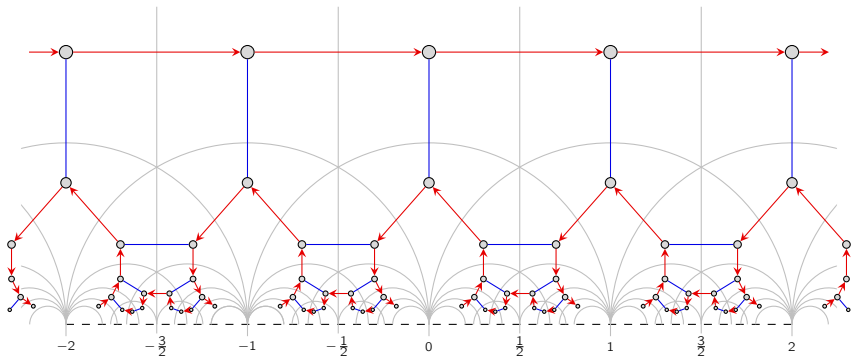$$A * B = \langle a, b \mid a^2 = 1, \, b^2 = 1 \rangle \cong D_\infty$$

Free products

The free product $C_3 * C_2$ is isomorphic to the projective linear group

$$\mathsf{PSL}_2(\mathbb{Z}) = \mathsf{SL}_2(\mathbb{Z})/\langle -I\rangle, \qquad \text{where } \mathsf{SL}_2(\mathbb{Z}) = \left\langle \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{S}, \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{T} \right\rangle.$$

This is in no way obvious from the generators that we've seen, which represent

$$S \colon z \longmapsto \frac{0z-1}{z+0} = -\frac{1}{z}, \qquad \text{and} \qquad T \colon z \longmapsto \frac{z+1}{0z+1} = z+1.$$
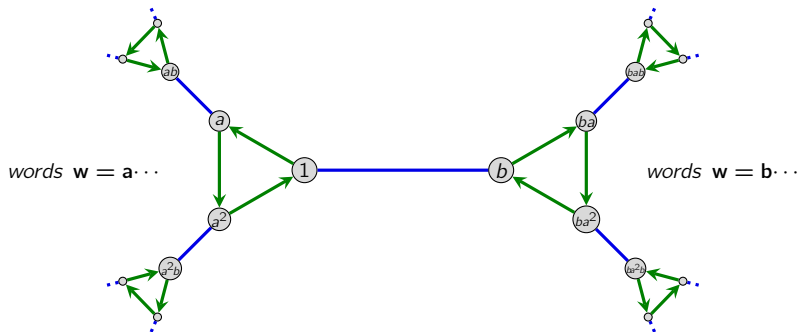
Free products

Let's see why the free product $C_3 * C_2$ is isomorphic to the projective linear group

$$\mathsf{PSL}_2(\mathbb{Z}) = \mathsf{SL}_2(\mathbb{Z})/\langle -I \rangle.$$

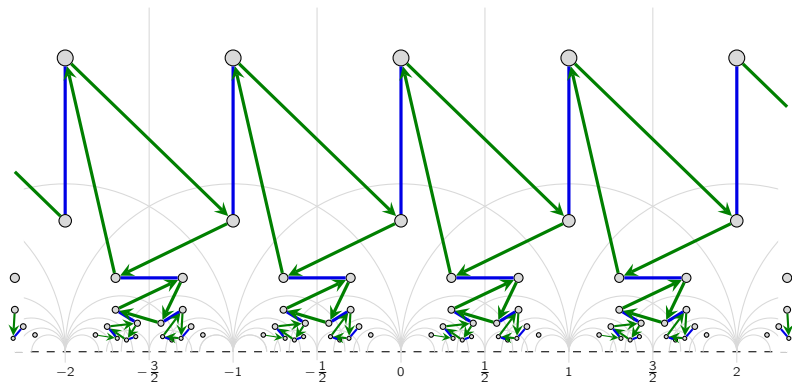Elements of $\mathsf{PSL}_2(\mathbb{Z})$ are cosets of $\langle -I \rangle = \pm I$. Let

$$\mathsf{SL}_2(\mathbb{Z}) = \langle S, T \mid S^2 = (ST)^6 = I \rangle, \qquad S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad ST = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix},$$

Then $\mathsf{PSL}_2(\mathbb{Z}) \cong \langle A, B \rangle$, where $A = \pm ST$ and $B = \pm S$.

words $\mathbf{w} = \mathbf{a} \cdots$ words $\mathbf{w} = \mathbf{b} \cdots$

Free products

A Cayley graph of $\mathsf{PSL}_2(\mathbb{Z}) = \langle A, B \mid A^3 = B^2 = 1 \rangle \cong C_3 * C_2$:

To verify $\mathsf{PSL}_2(\mathbb{Z}) \cong C_3 * C_2$, it suffices to show that we can't nontrivially write

$$I = A^{i_1} B^{j_1} A^{i_2} B^{j_2} \cdots A^{i_{m-1}} B^{j_{m-1}} A^{i_m}, \qquad i_k \in \{0, 1, 2\}, \quad j_k \in \{0, 1\}.$$

This will be left as HW.
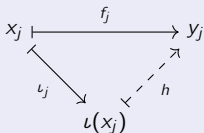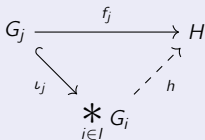
Free products

Definition

The free product of a family $G_i = \langle S_i \mid R_i \rangle$ of groups is

$$\underset{i \in I}{\LARGE *}\, G_i = \Big\langle \bigsqcup_{i \in I} S_i \;\Big|\; \bigsqcup_{i \in I} R_i \Big\rangle, \qquad \text{where } \iota_j \colon G_j \hookrightarrow \underset{i \in I}{\LARGE *}\, G_i, \qquad \iota_j(x_j) = x_j,$$

Exercise

The coproduct of $\big\{ G_i \mid i \in I \big\}$ in **Grp** is their free product.

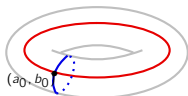That is, given any H and $\big\{ f_j \colon G_j \to H \mid j \in I \big\}$, there is a unique $h \colon \underset{i}{\LARGE *}\, G_i \to H$ such that $f_j = \iota_j \circ h$ for all $j \in I$.

Fiber coproducts in **Grp**: free products with amalgamation

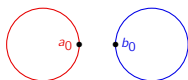Suppose A and B are disjoint circles. Gluing them at a point is called their wedge sum.

Product	*Coproduct*	*Coproduct w/ amalgamation*
A $\times$ **B**: "*Cartesian product*"	**A** $\sqcup$ **B**: "*disjoint union*"	**A** $\vee$ **B**: "*wedge sum*"

In general, we can identify or "glue" two objects along a common subset. Gluing two disks along their boundaries gives a sphere.

Suppose $A \trianglelefteq G_i$ for $i = 1, 2$, with embeddings $\alpha_i \colon A \hookrightarrow G_i$.

Goal: Take the the coproduct of G_1 with G_2, and "identify" the common subgroup A.

We can "force" $\alpha_1(a) \in G_1$ and $\alpha_2(a) \in G_2$ (in $G_1 * G_2$) to be the same by adding relations
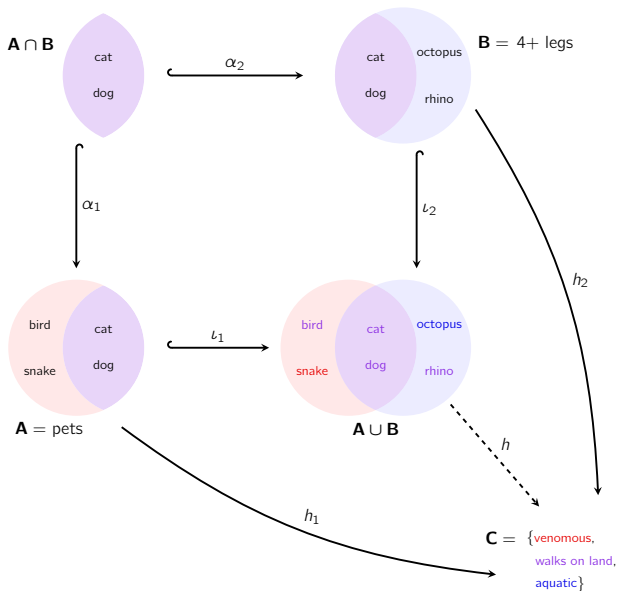
$$\alpha_1(a)\alpha_2(a)^{-1} = 1, \qquad \text{for all } a \in A,$$

and then quotient $A * B$ by the smallest normal subgroup N that contains these relators.

The group $G_1 *_A G_2 := (G_1 * G_2)/N$ is the free product of G_1 and G_2 amalgamated at A.

Fiber coproducts in **Set**: unions

Fiber coproducts in **Grp**: free products with amalgamation

$G_1 * G_2$ is the smallest group in which both G_1 and G_2 embeds into "independently."

I.e., for any other H with this property, those embeddings factor through via $G_1 * G_2 \to H$.

For $i = 1, 2$, let $\iota_i \colon G_i \to (G_1 * G_2)/N = G_1 *_A G_2$ be the map $\iota_i \colon g_i \mapsto g_i N$.
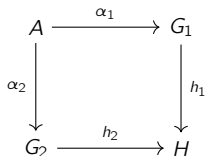
$$
\begin{array}{ccc}
A & \xrightarrow{\ \alpha_1\ } & G_1 \\
{\scriptstyle \alpha_2}\big\downarrow & & \big\downarrow{\scriptstyle \iota_1} \\
G_2 & \xrightarrow{\ \iota_2\ } & (G_1 * G_2)/N
\end{array}
\tag{2}
$$

$G_1 *_A G_2$ is the smallest group in which both G_1 and G_2 embeds into "independently," while keeping A identified.

The central product, e.g., $DQ_8 \cong D_4 \circ C_4 \cong Q_8 \circ C_4$, is a *direct product with amalgamation*.

Fiber coproducts in **Grp**: free products with amalgamation

Suppose G_1 and G_2 embed into H while keeping A identified:

$$
\begin{array}{ccc}
A & \xrightarrow{\ \alpha_1\ } & G_1 \\
\downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle h_1} \\
G_2 & \xrightarrow{\ h_2\ } & H
\end{array}
$$

Then $\exists! h \colon G_1 *_A G_2 \longrightarrow H$ that makes the following diagram commute:
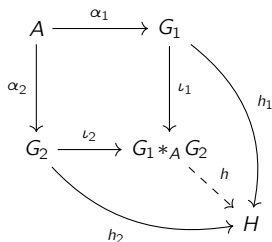
Fiber coproducts in a general category

Definition

Let $A, B_1, B_2 \in \mathsf{Ob}(\mathcal{C})$ and $\alpha_i \in \mathsf{Hom}_{\mathcal{C}}(A, B_i)$ for $i = 1, 2$. A fiber coproduct (or **pushout**) for them is a commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{\ \alpha_1\ } & B_1 \\
\downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \iota_1} \\
B_2 & \xrightarrow{\ \iota_2\ } & C
\end{array}$$

satisfying the following couniversal property:

> For any $D \in \mathsf{Ob}(\mathcal{C})$ and $h_i \in \mathsf{Hom}_{\mathcal{C}}(B_i, D)$ such that if $h_1 \circ \alpha_1 = h_2 \circ \alpha_2$, there exists a unique $h \in \mathsf{Hom}_{\mathcal{C}}(C, D)$ such that $h \circ \iota_i = h_i$.

Fiber coproducts (pushouts)

Proposition

Pushouts are unique up to equivalence.

Proof

Suppose we have two pushouts for A, B_1, B_2:

$$
\begin{array}{ccc}
A & \xrightarrow{\alpha_1} & B_1 \\
\downarrow{\alpha_2} & & \downarrow{\iota_1} \\
B_2 & \xrightarrow{\iota_2} & C
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & \xrightarrow{\alpha_1} & B_1 \\
\downarrow{\alpha_2} & & \downarrow{\delta_1} \\
B_2 & \xrightarrow{\delta_2} & D
\end{array}
$$

By the co-universal property, we have $h \in \mathsf{Hom}_{\mathcal{C}}(C, D)$ and $g \in \mathsf{Hom}_{\mathcal{C}}(D, C)$ such that:
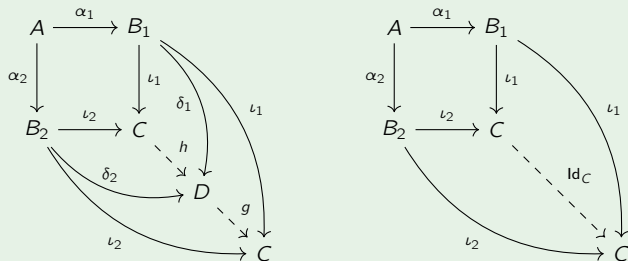
Fiber coproducts (pushouts)

Proposition

Pushouts are unique up to equivalence.

Proof (cont.)

We can "stack" these diagrams to get:

By uniqueness from the co-universal property, $g \circ h = \mathsf{Id}_C$.

Stacking them the other way gives $h \circ g = \mathsf{Id}_D$.

Therefore, h and g are inverse isomorphisms, and hence $C \cong D$. □

Fiber coproducts (pushouts)

In **Set** and **Top**, pushouts are ordinary unions:

$$
\begin{array}{ccc}
Y \cap Z & \overset{\alpha_Z}{\hookrightarrow} & Z \\
{\scriptstyle \alpha_Y} \downarrow & & \downarrow {\scriptstyle \iota_Z} \\
Y & \underset{\iota_Y}{\hookrightarrow} & Y \cup Z
\end{array}
$$

Siefert van-Kampen theorem

The functor $\pi_1 \colon \textbf{Top} \to \textbf{Grp}$ preserves pushouts.

$$
\begin{array}{ccc}
\mathcal{F}(Y \cap Z) & \overset{\alpha_Z^*}{\longrightarrow} & \mathcal{F}(Z) \\
{\scriptstyle \alpha_Y^*} \downarrow & & \downarrow {\scriptstyle \iota_Z^*} \\
\mathcal{F}(Y) & \underset{\iota_Y^*}{\longrightarrow} & \mathcal{F}(Y \cup Z)
\end{array}
\qquad
\begin{array}{ccc}
\pi_1(Y \cap Z) & \overset{\alpha_Z^*}{\longrightarrow} & \pi_1(Z) \\
{\scriptstyle \alpha_Y^*} \downarrow & & \downarrow {\scriptstyle \iota_Z^*} \\
\pi_1(Y) & \underset{\iota_Y^*}{\longrightarrow} & \pi_1(Y \cup Z)
\end{array}
$$

The Siefert van-Kampen theorem

$\pi_1(Y \cap Z) = \langle aba^{-1}b^{-1} \rangle \cong \mathbb{Z}$

$\pi_1(Z) = \langle 1 \rangle$

Y ∩ Z

α_Z

α_Y

Z = D²

ι_Z

ι_Y

Y = T² − D²

X = Y ∪ Z = T²
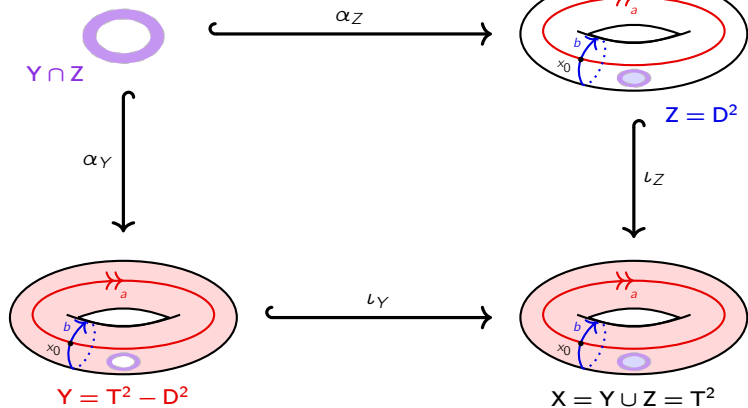
$\pi_1(Y) = \langle a, b \mid \, \rangle$

$\pi_1(Y \cup Z) = \pi_1(Y) *_{\pi_1(Y \cap Z)} \pi_1(Z) \cong \langle a, b \mid \, \rangle *_{\langle aba^{-1}b^{-1} \rangle} \langle 1 \rangle$

$= \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle \cong \mathbb{Z} \times \mathbb{Z}$

Fiber products / pullbacks (HW)

Definition

Let $A_1, A_2, B \in \mathrm{Ob}(\mathcal{C})$ and $\alpha_i \in \mathrm{Hom}_{\mathcal{C}}(A_i, B)$ for $i = 1, 2$. A fiber product (or **pullback**) for them is a commutative diagram

$$
\begin{array}{ccc}
P & \xrightarrow{\pi_1} & A_1 \\
\downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \alpha_1} \\
A_2 & \xrightarrow{\alpha_2} & B
\end{array}
$$

satisfying the following universal property:

For any $Q \in \mathrm{Ob}(\mathcal{C})$ and $h_i \in \mathrm{Hom}_{\mathcal{C}}(Q, A_i)$ such that if $\alpha_1 \circ h_1 = \alpha_2 \circ h_2$, there exists a unique $h \in \mathrm{Hom}_{\mathcal{C}}(Q, P)$ such that $h_i = \pi_i \circ h$.