**Clemson University**
**School of Mathematical & Statistical Sciences**

# SYLLABUS
## MATH 8530-001, Linear Algebra / Matrix Analysis
## Fall 2023
## Martin E-004, MWF 12:20–1:10pm

**Instructor:** Matthew Macauley, Martin Hall O-325, macaule@clemson.edu, 656-1838 (no voicemail)

**Course Description:** This course will be a comprehensive survey of finite dimensional vector spaces and linear mappings at the graduate level. The goals are two-fold: (i) to prepare students for the linear algebra part of the Algebra Prelim, and (ii) to build up a solid linear algebra foundation necessary for other classes and research, as linear algebra is a beautiful subject that appears in nearly all areas of mathematics, *especially* in applied fields such as statistics, operations research, and computational mathematics. Some advanced linear algebra classes, especially those geared to engineers and often called "matrix analysis," focus on computations involving matrices and under-emphasize the theory of vector spaces. Yet many other treatments take the opposite approach by presenting the theory of modules, which is too far detached from most applications that an applied mathematician, statistician, or optimizer will need. I will try to take a "Goldilocks" approach, and emphasize the theory of vector spaces (not modules), while being grounded in applications. The book that I will loosely follow, despite being quite theoretical, was written by a foremost expert on computational PDEs, containing that material that thought was most important for his applied mathematics graduate students to learn over the course of a year-long class.

**Prerequisite:** Any undergraduate linear algebra class (Math 3110 at Clemson).

**Modality:** This is an in-person class, but there is considerable uncertainty due to things like COVID. We all must be prepared to shift to an online modality with little notice, and I will email a Zoom link that will be used if this happens. Though I plan to follow whatever the current university modality policy is, there is a strong chance that on at least one day, I will be sick (nearly impossible to avoid with a preschooler at home), traveling due to a conference, or we will have a weather event like a hurricane, tornado, or snowstorm. If this happens, we will have class online (could be either synchronous or asynchronous), rather than it being canceled or taught by a substitute, and I will provide details ahead of time.

**Communication Strategy:** I generally prefer to communicate via emails rather than Canvas messages. I encourage all of us to be on a first-name basis.

All of my email addresses (e.g., **macaule@clemson.edu**) go to the same gmail inbox, which I check multiple times a day. Though I have the gmail iPhone app, I don't get email push notifications. Also, I usually don't check email on Saturdays.

I will typically be in my office at least MWF, from roughly 8:15am–5:00pm. If you need to reach me immediately during the week, feel free to call my office phone, at (864) 656-1838. If I'm not there, please send an email rather than leaving a voicemail.

*If you send me an email and do not get a reply by the time you go to bed, please re-send it, as that is either my mistake, or I haven't yet gotten to it, in which case that will bump it up to the top of my inbox. Just click "Reply" and "Send"; no need to explain.*

**Happy Hour:** It has been my experience that evening Zoom office hours are much more frequently attended than (pre-COVID) on-campus office hours ever were. As such, twice a week at 7pm (dates TBA), you are invited to join me and your classmates on Zoom for an "Adult Beverage" – for me this means drinks like **LaCroix** or **Kombucha**, which are *very unpopular* among kids.[1], company, and office hours. I'll stick around to answer questions as long as there are some. However, unless I say otherwise in class, *I will only show up if at least one person RSVPs, either in class or by sending me an email before 4:00pm, saying they will attend.* If no one is there by 7:05pm, I will log off.

I am also available to meet by appointment, if desired, either over Zoom or in person, in my office. To make an appointment, email me and include block(s) oftime in which you are available. Please let me know in advance if you want any Zoom meeting to be private, like if you want to discuss your grade. In that case, I will use a different Zoom meeting or a breakout room.

**Useful websites:**

*Course webpage*: `http://www.math.clemson.edu/~macaule/classes/f23_math8530/` (all relevant links posted here)

*Canvas*: `https://www.clemson.edu/canvas/` (will be used very sparingly, if at all)

*Gradescope*: `https://www.gradescope.com/` (click "Log in", then "School Credentials")

**Texts:** There is no need to buy a book for this class. In lecture, I will roughly follow the first 9 chapters of Peter Lax's book *Linear Algebra and its Applications*, though I will use Paul Halmos' classic *Finite Dimensional Vector Spaces* when we cover determinants, and I will also add some additional topics. I will also occasionally draw from Gil Strang's book. For a nice freely available book that covers the same material we are, I recommend:

J. Gallier and J. Quaintance, *Algebra, Topology, Differential Calculus, and Optimization Theory for Computer Science and Machine Learning*, Part I, 2019, freely available online at `http://www.cis.upenn.edu/~jean/gbooks/geomath.html`

**Schedule:** I taught Math 8530 online both semesters of the 2020–2021 academic year, and I broke the material into 49 lectures, most of which I have already recorded and posted to YouTube. Because of this, I will adhere to a somewhat strict weekly schedule. If we start to fall behind, or don't get to all of the material in a given week, I may ask students to catch up by watching part of a video on their own time. On the course webpage, I will post:

- A complete *Course Calendar*, with topics by week that I plan to cover.
- A cumulative *Weekly Summary* of what we did cover in class, what videos to watch online, etc.

**Required technology:**

A computer and reliable internet connection.

A free Zoom account, and a video camera that allows you to be recorded over Zoom.

A smartphone scanning app. There are many free apps, such as CamScanner or Adobe Scan. If you do not have a smartphone, a traditional scanner will suffice, but a smartphone app is prefered.

Access to a program that runs LaTeX. I recommend `overleaf.com`.

An ad blocker, in case there are videos to watch. I strongly recommend the free AdBlock extension.

[1] Or so I thought when I wrote this. My 4-year-old loves both, and you'll likely see her drop by and mooch off of my drink during office hours.

**Zoom Info:** Please turn your camera on for all Zoom meetings, if there are any.

There will be one common Zoom URL for all online classes (if needed), and another for all Happy Hours, which may also be attended by my other (undergradate) class. These URLs will be emailed to the class list.

If your internet goes out during a Zoom meeting, Zoom should automatically reconnect when it comes back. If mine goes out during an online class, please stick around, and consider this to be an unexpected 1–2 minute break from class.

**Homework:** There will be 14 weekly homework assignments, usually consisting of 5 problems. Though it can be a difficult habit to form, *I strongly encourage students try to complete one problem per day, rather than save them all for the last 24–48 hours.*

*Submitted assignments must be typeset with LaTeX.* The pdf and tex files will be posted on the course webpage.

Working together is encouraged, but everyone must do their own work, and *collaborators must be cited.* Outside sources, such as webpages, must also be cited as well, if they were instrumental to a solution.

Homework assignments are due at 11:59pm EST, and will be submitted either through Canvas or Gradescope. Assignments can be submitted multiple times; only the last submission will be graded. There is a 3-hour grace period for deadlines, meaning that there is no penalty for assignments submitted within 3 hours of the deadline.

Assignments submitted after this period will not be accepted, but I will drop everybody's lowest score. This policy is intended to be an alternative to giving extensions due to unexpected circumstances such as an illness.

**Attendance:** Please make an effort to attend all classes, and to be on time. I will try to show up 10 minutes early to all classes. In the unlikely scenario that I am absent 5 minutes after class has started, check your email. If you have not heard from me 10 minutes after class has begun, you may assume that class has been canceled.

If you cannot make class, due to illness, quarantine, or some other reason, please let me know as a courtesy.

**Exams:** There will be 2 in-class midterm exams, and one final exam. I will drop either your lowest midterm or one-half the final exam's weight. All exams will be closed notes and closed book.

**Grading:** Your final grade will be computed as follows:

| | |
|---|---|
| Homework | 25% |
| Midterm 1 | 25% |
| Midterm 2 | 25% |
| Final Exam | 50% |

I will drop either your lowest midterm grade, OR half of the weight of the final exam; whichever is lowest. Also, if you get an unambiguous **A** on the final exam, then you will get an **A** in the course. The same goes for **A-**. Basically, if you can do well on a cumultative 3-hour final exam, then you deserve a good grade (**A** or **A-**) in this class.

I do *not* grade using arbitrary round number cut-offs. I generally err on the side of having difficult exams (e.g., sometimes the median score is in the 50–60% range). This spreads out the distribution, and the scores generally fall into visually clear distinct "clusters" with large gaps between them. Roughly speaking, an **A** is the "very good" cluster, a **B** is statistically significantly below that, and a **C** is for outliers on the low end. I will use **+** and **-** grades for borderline cases, or for a bi-modal or very wide cluster. Though past performance is not an indicator of future results, I have never given fewer than 50% **A**s in a graduate class.

Because of the aforementioned policies, the automatically calculated numeric grade that you see in Canvas, which is the average of every weighted graded submission, is NOT an accurate indicator of your grade. At any point in time during the class, I would be happy to give you a ballpark estimate of how you are doing.

**Key Dates**

| | |
|---|---|
| Aug 23 (Wed) | Classes begin |
| Aug 29 (Tue) | Last day to register or add a class |
| Sep 4 (Mon) | Labor Day (first time ever Clemson has observed it!) |
| Sep 6 (Wed) | Last day to drop a class or withdraw from the University w/o a W grade |
| Nov 1 (Wed) | Last day to drop a class or withdraw from the University w/o final grades |
| Oct 16–17 (M–Tu) | Fall break |
| Nov 22–24 (W–F) | Thanksgiving break |
| Dec 8 (Fri) | Last day of class |
| Dec 11 (Mon) | Final Exam, 8–10:30am |

**Student Learning Outcomes:** Upon successful completion of MATH 8530, students will be able to

Demonstrate a solid understanding of the theory of vector spaces and linear maps.

Abstract skills and knowledge learned in undergraduate linear algebra to a more general setting.

Apply concepts from linear algebra to problems in applied mathematics and statistics.

Read, write, and critique rigorous mathematical proofs on topics in linear algebra.

Develop good mathematical writing skills. Important aspects of this are *accuracy*, *clarity*, and *conciseness*.

**"No exceptions":** In any class syllabus, no matter how they are worded, policies and phrases like "no exceptions", "no make-ups", etc. are *never* actually what they sound, and this is especially true this semester. Things happen, from natural distasters (hurricanes, tornados), to human disasters (9/11, school shootings), to personal and family tragedies, to health emergencies (COVID, auto accidents, hospitalizations). This does not mean that any exception or extension will be granted, but I will do my best to be reasonable, fair, and accommodating.

**Make-Up Policy:** I will drop your lowest midterm, which means that if you miss a midterm, then your final exam grade will replace it. The homework deadlines will not be extended for individual students, and assigned homework must be turned in by the deadline. PLAN AHEAD: If you submit assignments minutes before the deadline, you take the risk of bad luck, e.g., a power outage, computer freeze or crash, personal emergency, zombie attack, etc., that could make you miss the deadline.

By default, any exam that was scheduled at the time of a class cancellation due to power outage / inclement weather will be given at the next class meeting. Any extension or postponement of assignments or exams must be granted by me via email or Canvas within 24 hours of the weather-related cancellation.

**Special Accommodations:** Clemson University values the diversity of our student body as a strength and a critical component of our dynamic community. Students with disabilities or temporary injuries/conditions may require accommodations due to barriers in the structure of facilities, course design, technology used for curricular purposes, or other campus resources. Students who experience a barrier to full access to a class should let the instructor know, and make an appointment to meet with a staff member in Student Accessibility Services as soon as possible. You can make an appointment by calling 864-656-6848 or by emailing studentaccess@lists.clemson.edu. Students who receive Academic Access Letters are strongly encouraged to request, obtain and present these to their instructors as early in the semester as possible so that accommodations can be made in a timely manner. It is the student's responsibility to follow this process each semester. You can access further information here: `http://www.clemson.edu/campus-life/campus-services/sds/`.

**Mental health:** Your mental health is important to me, and I am always available to talk. Please don't hesitate to reach out. We're in this together, and all of us are strugling in some regards, myself included.

**Title IX Policy:** Clemson University is committed to a policy of equal opportunity for all persons and does not discriminate on the basis of race, color, religion, sex, sexual orientation, gender, pregnancy, national origin, age, disability, veteran's status, genetic information or protected activity (e.g., opposition to prohibited discrimination or participation in any complaint process, etc.) in employment, educational programs and activities, admissions and financial aid. This includes a prohibition against sexual harassment and sexual violence as mandated by Title IX of the Education Amendments of 1972.

The University is committed to combatting sexual discrimination including sexual harassment and sexual violence. As a result, you should know that University faculty and staff members who work directly with students are required to report any instances of sexual harassment and sexual violence, to the University's Title IX Coordinator. What this means is that as your professor, I am required to report any incidents of sexual harassment, sexual violence or misconduct, stalking, domestic and/or relationship violence that are directly reported to me, or of which I am somehow made aware.

There are two important exceptions to this requirement about which you should be aware:

Confidential Resources and facilitators of sexual awareness programs such as "Take Back the Night and Aspire to be Well" when acting in those capacities, are not required to report incidents of sexual discrimination.

Another important exception to the reporting requirement exists for academic work. Disclosures about sexual harassment, sexual violence, stalking, domestic and/or relationship violence that are shared as part of an academic project, a research project, classroom discussion, or course assignment, are not required to be disclosed to the University's Title IX Coordinator.

This policy is at `http://www.clemson.edu/campus-life/campus-services/access/title-ix/`. Alesia Smith is the Executive Director for Equity Compliance and the Title IX Coordinator. Her office is at 223 Holtzendorff Hall, phone number is 864.656.3181, and email address is alesias@clemson.edu.

**Academic Integrity:** As members of the Clemson University community, we have inherited Thomas Green Clemson's vision of this institution as a 'high seminary of learning'. Fundamental to this vision is a mutual commitment to truthfulness, honor, and responsibility, without which we cannot earn the trust and respect of others. Furthermore, we recognize that academic dishonesty detracts from the value of a Clemson degree. Therefore, we shall not tolerate lying, cheating, or stealing in any form.

**Copyright Statement:** Some of the materials in this course are possibly copyrighted. They are intended for use only by students registered and enrolled in this course and only for instructional activities associated with and for the duration of the course. They may not be retained in another medium or disseminated further. They are provided in compliance with the provisions of the Teach Act. Refer to the Use of Copyrighted Materials and "Fair Use Guidelines" policy on the Clemson University website for additional information: `http://clemson.libguides.com/copyright`.

**Topics:** We will cover the following topics, roughly in this order.

**Section 1**: Groups

- Vector spaces and subspaces
- Spanning, independence, bases, and dimension
- Application: solving linear ODEs
- Complementary subspaces and direct sums
- Direct products
- Finite vs. infinite products
- Congruence of subspaces and quotient spaces
- Linear functionals & the dual space
- Scalar product notation
- Double duals
- Annihilators

**Section 2**: Linear mappings [Chapters 3–4 of Lax]

- Range & nullspace
- Rank-nullity theorem
- Application to polynomial interpolation
- Application to average values of polynomials over intervals
- Application to systems of linear equations
- Application to ODEs: the method of undetermined coefficients
- Application to PDEs: numerial solutions to Laplace's equation (finite differences)
- Algebra of linear mappings
- Transposes (as mappings between dual spaces)
- How a choice of basis determines the matrix of a linear map
- 4 ways to multiply matrices (row-by-cols, by rows, by cols, col-by-rows)
- The matrix of the transpose map
- Column rank & row rank
- Change of basis & similar matrices

**Section 3**: Multilinear forms [Halmos' book + Strang's book + Chapter 5 and Appendix 4 of Lax]

- Signed volume
- Permutations & discriminant
- Multilinear forms
- Symmetric, skew-symmetric, and alternating $k$-linear forms

- The vector space of alternating $n$-linear forms is 1-dimensional
- Basis-free definition & universal property of the determinant
- The determinant of a matrix
- The trace of a matrix
- Laplace expansion, minors, and cofactors
- Cramer's rule
- Basis-free construction of the tensor product
- Universal property of the tensor product
- Tensors as linear maps
- Extending $\mathbb{R}$-vector spaces with tensors

**Section 4**: Spectral theory [chapter 6 of Lax + appendix 15 of Lax + supplimental]

- Eigenvectors and eigenvalues
- Distinct eigenvalues lead to linearly independent eigenvectors
- Diagonalizability
- The characteristic polynomial
- Determinant and trace in terms of eigenvalues
- Spectral mapping theorem: Eigenvalues of $A$ vs. $q(A)$.
- Cayley-Hamilton theorem
- Algebraic multiplicity vs. geometric multiplicity of eigenvalues
- Minimal polynomials and generalized eigenvectors
- Invariant subspaces and block-diagonal matrices
- Spectral theorem: there is always a full set of generalized eigenvectors
- Jordan canonical form
- Commuting maps and simulataneous diagonaliziability
- Application: systems of ODEs & matrix exponentials
- Cyclic subspaces
- Companion matrices and rational canonical form

**Section 5**: Inner product spaces [Chapter 7 of Lax]

- Review of Euclidean structure (length, dot product, orthogonality, angles)
- Real inner product spaces
- Cauchy-Schwarz inequality
- Triangle inequality & Pythagorean theorem
- Orthogonality
- Orthonormal bases
- Orthogonal polynomials and generalized Fourier series
- Gram-Schmidt process & QR-factorization
- Identification of a space with its dual
- Adjoints
- Orthogonal complement & projections
- Application: least squares
- Isometries & orthogonal maps
- The Frobenius and induced norms of a linear map
- The subset of invertible maps is open
- Basic analysis review (convergence, Cauchy sequences, completeness, local compactness)
- An inner product spaces is locally compact iff it is finite-dimensional

- Sesquilinear forms and complex inner product spaces
- Unitary maps
- Complex Fourier series

**Section 6**: Self-adjoint mappings [chapter 8 of Lax]

- Decomposition of a linear map into a self-adjoint plus an anti-self-adjoint map
- Motivation: 2nd order Taylor approximations & the Hessian
- Quadratic forms and conic sections
- Unitarily diagonalization
- Projections onto eigenspaces and spectral resolutions
- Self-adjoint commuting maps have a common spectral resolution
- Anti-self-adjoint maps have purely imaginary eigenvalues and a full set of orthonormal vectors
- Normal maps
- The Rayleigh quotient & its critical points
- Minmax principle for the eigenvalues of a self-adjoint map
- Application to numerical linear algebra: 2nd order Taylor approximation of eigenvalues
- Application to ODEs: Sturm-Liouville theory and generalized Fourier series

**Section 7**: Positive(-definite) mappings [Chapter 10 of Lax]

- Basic properties of positive and non-negative mappings
- The set of positive maps is open
- Non-standard inner products
- The generalized Reyleigh quotient
- Gram matrices
- Properties of $A^*A$.
- Schur's theorem of positive matrices
- The non-negative square root of a positive matrix
- Polar decomposition
- Singular value decomposition (SVD)
- Right, left, and pseudo-inverses
- Applications of SVD
- A partial order on on the set of self-adjoint maps
- Symmetrized products
- Monotone matrix functions (MMFs)
- A functional analysis characterization of all MMFs