Read: Lax, Chapter 1, pages 1–11.

1. Let X be a vector space over a field K. Let 0 be the zero element of K and $\mathbf{0}$ the zero-element of X. Using only the definitions of a group, a vector space, and a field, carefully prove each of the following:
   (a) The identity element e of a group is unique.
   (b) In any group G, the inverse of $g \in G$ is unique.
   (c) $0x = \mathbf{0}$ for every $x \in X$;
   (d) $k\mathbf{0} = \mathbf{0}$ for every $k \in K$;
   (e) For every $k \in K$ and $x \in X$, if $kx = \mathbf{0}$, then $k = 0$ or $x = \mathbf{0}$.

2. Let X be a vector space over K, and let S be a linearly independent subset of X. The following is called the *exchange property*: For any nonzero $x_0 \in \mathrm{Span}(S)$, there exists $x_1 \in S$ such that $S' = (S \setminus \{x_1\}) \cup \{x_0\}$ is a basis for $\mathrm{Span}(S)$.
   (a) Prove the exchange property.
   (b) Suppose that B is a size-n basis for X, and let B' be another basis. Use the exchange property to show that $|B'| = n$.

3. Let X_1, X_2 be a subspace of a finite-dimensional K-vector space X. Show that $\dim(X_1 \times X_2) = \dim(X_1 \oplus X_2)$.

4. If Y is a subspace of X, then two vectors $x_1, x_2 \in X$ are *congruent modulo* Y, denoted $x_1 \equiv x_2 \bmod Y$, if $x_1 - x_2 \in Y$. This is an equivalence relation; denote the equivalence class containing $x \in X$ by $\{x\}$, and let X/Y denote the set of equivalence classes. We can make X/Y into a vector space by defining addition and scalar multiplication as follows:
$$\{x\} + \{z\} := \{x + z\}, \qquad a\{x\} := \{ax\}.$$
   (a) Show that these operations are *well-defined*. That is, they do not depend on the choice of congruence class representatives.
   (b) Prove that for any $x \in X$, the following two sets are equal as subsets of X:
$$\{x\} = \{x + y \mid y \in Y\}.$$
   This motivates the alternative "coset notation" of $x + Y$ for the equivalence class of x modulo Y. Show how to add and scalar multiply in this notation by computing:
$$(x + Y) + (z + Y), \qquad \text{and} \qquad a(x + Y), \quad x, z \in X,\ a \in K.$$
   (c) Assume $\dim X < \infty$. Show that X is isomorphic to $Y \times X/Y$ by defining an explicit map and showing that it is linear and a bijection.

5. Let Z be a subspace of X.
   (a) Show that if $Z \le Y \le X$, then Y/Z is a subspace of X/Z.
   (b) Show that *every* subspace of X/Z arises in this manner.