Read: Lax, Chapter 2, pages 13–18.

1. The *characteristic* of a field K, denoted $\operatorname{char} K$, is the smallest positive integer n for which $n1 := \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = 0$, or zero if no such n exists.

   (a) Show that if K and L are fields with $K \leq L$, then L is a vector space over K.

   (b) Show that the characteristic of a field is either zero or prime.

   (c) Show that if K is a finite field, then $|K|$ is a prime power.

   (d) Show that if K and L are finite fields with $K \subset L$ and $|K| = p^m$ and $|L| = p^n$, then m divides n.

2. Let X be a vector space over a field K and let X' be the the set of linear functions from X to K, also known as the *dual space* of X.

   (a) Let $x_1, \dots, x_n$ be a basis for X. Define $\ell_j \in X'$ by $\ell_j(x_i) = \delta_{ij}$. Show that $\ell_1, \dots, \ell_n$ is a basis for X'; it is called the *dual basis* of $x_1, \dots, x_n$.

   (b) Find the dual basis of $x_1 = (1, -1, 3)$, $x_2 = (0, 1, -1)$, and $x_3 = (0, 3, -2)$ in $X = \mathbb{R}^3$.

   (c) Express the scalar function $f \in X'$, where $f(x, y, z) = 2x - y + 3z$ as a linear combination of the dual basis, ℓ_1, ℓ_2, ℓ_3, from Part (b).

3. Let S be a subset of X. The *annihilator* of S is the set

$$S^{\perp} = \{\ell \in X' \mid \ell(s) = 0 \text{ for all } s \in S\} .$$

   (a) Show that $\operatorname{span}(S)$ is the intersection of all subspaces T_α of X that contain S:

$$\operatorname{span}(S) = \bigcap_{S \subseteq T_\alpha \leq X} T_\alpha,$$

   making it well-founded to speak of the "*smallest subpace of X that contains S*."

   (b) Show that $S^{\perp}$ is a subspace of X', and that $S^{\perp} = \operatorname{span}(S)^{\perp}$.

4. Let $\mathcal{P}_2$ be the vector space of all polynomials $p(x) = a_0 + a_1 x + a_2 x^2$ over $\mathbb{R}$, with degree ≤ 2. Let ξ_1, ξ_2, ξ_3 be distinct real numbers, and define

$$\ell_j \colon \mathcal{P}_2 \longrightarrow \mathbb{R}, \qquad \ell_j(p) = p(\xi_j) \quad \text{for} \quad j = 1, 2, 3 .$$

   (a) Show that ℓ_1, ℓ_2, ℓ_3 is a basis for the dual space $\mathcal{P}_2'$.

   (b) Find polynomials $p_1(x), p_2(x), p_3(x)$ in $\mathcal{P}_2$ of which ℓ_1, ℓ_2, ℓ_3 is the dual basis in $\mathcal{P}_2'$.

5. Let X be a vector space with basis $x_1, \dots, x_4$, and W the subspace spanned by $x_1 - x_3 + 2x_4$ and $2x_1 + 3x_2 + x_3 + x_4$. Find a basis for the annihilator of W. Write you answer in terms of the dual basis vectors $\ell_1, \dots, \ell_4$ of X'.