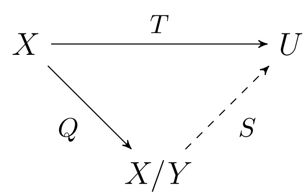
Read: Lax, Chapter 3, pages 19–25, 28–31.

1. Let $T\colon X \to U$ be a linear map. Prove the following:

   (a) The image of a subspace of X is a subspace of U.

   (b) The inverse image of a subspace of U is a subspace of X.

2. Let X and U be vector spaces, and suppose that Y is a subspace of X. Let $Q\colon X \to X/Y$ be the canonical quotient map sending $x \overset{Q}{\longmapsto} \{x\}$, and let $T\colon X \to U$ be a linear map. State and prove a condition that is both necessary and sufficient for the existence of a unique linear map $S\colon X/Y \to U$ such that $T = S \circ Q$. When this happens, the map T is said to *factor through* the quotient space, as shown by the following commutative diagram:



3. Suppose $T\colon X \to X$ is a linear map of rank 1.

   (a) Show that there exists $c \in K$ such that $T^2 = cT$.

   (b) Show that if $c \neq 1$, then $I - T$ has an inverse.

4. Suppose that $S, T\colon X \to X$ are linear maps.

   (a) Show that $\operatorname{rank}(S+T) \leq \operatorname{rank}(S) + \operatorname{rank}(T)$.

   (b) Show that $\operatorname{rank}(ST) \leq \operatorname{rank}(S)$.

   (c) Show that $\dim(N_{ST}) \leq \dim N_S + \dim N_T$.

   For each of these, without using matrices, give an explicit example of where equality holds, and another where it does not.

5. Let $T : X \to X$ be linear, with $\dim X = n$.

   (a) Prove that if $T^2 = T$, then $X = R_T \oplus N_T$.

   (b) Show by example that if $T^2 \neq T$, then $X = R_T \oplus N_T$ need not hold.

   (c) Prove that $N_{T^n} = N_{T^{n+1}}$ and $R_{T^n} = R_{T^{n+1}}$.

   (d) Prove that $X = R_{T^n} \oplus N_{T^n}$.

   (e) Show there exists a linear map $S : X \to X$ such that $ST = TS$ and $ST^{n+1} = T^n$.