*Read*: Lax, Chapter 5, pages 44–56.

1. Let $S_n$ denote the set of all permutations of $\{1, \dots, n\}$.

   (a) Prove that $\operatorname{sgn}(\pi_1 \circ \pi_2) = \operatorname{sgn}(\pi_1)\operatorname{sgn}(\pi_2)$.

   (b) Prove that $\operatorname{sgn}(\tau) = -1$ for all transpositions $\tau \in S_n$.

   (c) Let $\pi \in S_n$, and suppose that $\pi = \tau_k \circ \cdots \circ \tau_1 = \sigma_\ell \circ \cdots \circ \sigma_1$, where $\tau_i, \sigma_j \in S_n$ are transpositions. Prove that $k \equiv \ell \bmod 2$.

2. Let $f$ be a *non-degenerate* symmetric bilinear form over an $n$-dimensional vector space $X$. That is, for all nonzero $x \in X$, there is some $y \in X$ for which $f(x, y) \neq 0$. Consequently, fixing any nonzero $x \in X$ defines a nonzero dual vector

$$f(x, -) \in X', \qquad f(x, -) \colon y \longmapsto f(x, y).$$

   (a) Prove that the map $L_f \colon X \to X'$ given by $L_f \colon x \mapsto f(x, -)$ is an isomorphism.

   (b) Let $x_1, \dots, x_n$ be a basis for $X$. Express the dual basis $\ell_1, \dots, \ell_n$ in this form. That is, find $g$ for which $L_g \colon x_i \mapsto \ell_i$.

   (c) Show how to construct another basis $y_1, \dots, y_n$ such that $f(x_i, y_j) = \delta_{ij}$.

   (d) Conversely, prove that if $\mathcal{B}_X = \{x_1, \dots, x_n\}$ and $\mathcal{B}_Y = \{y_1, \dots, y_n\}$ are sets of vectors in $X$ with $f(x_i, y_j) = \delta_{ij}$, then $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for $X$.

3. Let $f$ be a non-degenerate symmetric bilinear form over an $n$-dimensional vector space, where $1 + 1 \neq 0$.

   (a) Show that there exists $x_1 \in X$ with $f(x_1, x_1) \neq 0$.

   (b) Let $Z_1$ be the nullspace of $f(x_1, -)$. Show that $f$ restricted to $Z_1$ is non-degenerate.

   (c) Construct a basis $\{z_1, \dots, z_n\}$ for $X$ that satisfies $f(z_i, z_j) = \delta_{ij}$.

4. Let $f$ be a bilinear form over a vector space $X$ with basis $\{x_1, x_2\}$.

   (a) Assume $f$ is alternating. Determine a formula for $f(u, v)$ in terms of each $f(x_i, x_j)$ and the coefficients used to express $u$ and $v$ with this basis. [Pun intented!]

   (b) Repeat Part (a) but assume that $f$ is symmetric and $f(x, x) = 0$ for all $x \in X$.

5. Prove the following properties of the trace function:

   (a) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for all $m \times n$ matrices $A$ and $n \times m$ matrices $B$.

   (b) If $A$ is square, write down a formula for $\operatorname{tr}(AA^T)$ in terms of $a_{ij}$.