Read: Lax, Appendix 15, pages 363–366.

1. Let A be a 7×7 matrix over $\mathbb{C}$ with minimal polynomial $m(t) = (t-1)^3(t-2)^2$.
   (a) List all possible Jordan canonical forms of A up to similarity.
   (b) For each matrix from Part (a), find the rank of $(A-I)^k$ and $(A-2I)^k$, for $k \in \mathbb{N}$.

2. Let A be an $n \times n$ matrix over $\mathbb{C}$. The matrix A is *nilpotent* if $A^k = 0$ for some $k \in \mathbb{N}$.
   (a) Prove that if A is nilpotent, then $A^n = 0$.
   (b) Prove that if A is nilpotent, then there is some $r \in \mathbb{N}$ and positive integers $k_1 \geq \cdots \geq k_r$ with $k_1 + \cdots + k_r = n$ that determine A up to similarity.
   (c) Suppose A and B are 6×6 nilpotent matrices with the same minimal polynomial and $\dim N_A = \dim N_B$. Prove that A and B are similar. Show by example that this can fail for 7×7 matrices.

3. Let A and B be $n \times n$ matrices over $\mathbb{C}$. The matrix A is *idempotent* if $A^2 = A$.
   (a) Prove that if $A^k = A$ for some integer $k > 1$, then A is diagonalizable.
   (b) Prove that idempotent matrices are similar if and only if they have the same trace.
   (c) Prove that if A and B are idempotent and $B = UAV$ holds for some invertible maps $U, V \colon X \to X$, then A and B are similar.

4. Consider the matrices $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -3 & 0 \\ -3 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.
   (a) Decompose $\mathbb{R}^3$ into a direct sum of eigenspaces of each matrix.
   (b) Further decompose the 2-dimensional A-eigenspace as a direct sum of two 1-dimensonal B-eigenspaces, and vice-versa.
   (c) Write $\mathbb{R}^3$ as a direct sum of three 1-dimensional subspaces that are common eigenspaces of A and B, two different ways.
   (d) For each of your answers to Part (c), find a matrix P so that $P^{-1}AP = D_A$ and $P^{-1}BP = D_B$, where D_A and D_B are diagonal.

5. Let X be an n-dimensional vector space over $\mathbb{C}$, and let $A, B \colon X \to X$ be linear maps.
   (a) Prove that if $AB = BA$, then for any eigenvector v of A with eigenvalue λ, the vector Bv is an eigenvector of A for λ.
   (b) Suppose that A and B are both diagonalizable. Prove that $AB = BA$ if and only if they are *simultaneously diagonalizable*, i.e., there exists an invertible $n \times n$-matrix P such that both $P^{-1}AP$ and $P^{-1}BP$ are diagonal matrices.
   (c) Show that if $\{A_1, \ldots, A_k \mid A_i \colon X \to X\}$ is a set of pairwise commuting maps, then there is a nonzero $x \in X$ that is an eigenvector of every A_i.