Read: Lax, Chapter 7, pages 89–100.

1. Let X be a complex inner product space, and $A\colon X \to X$.
   (a) Show that if $\langle Ax, x\rangle = 0$ for all $x \in X$, then $A = 0$.
   (b) Give an explicit example of how the previous part fails if X is a real inner product space.
   (c) Show that A if self-adjoint if and only if $\langle Ax, x\rangle \in \mathbb{R}$ for all $x \in X$.
   (d) Show that on a real inner product space, A is self-adjoint and $\langle Ax, x\rangle = 0$ for all $x \in X$, then $A = 0$.

2. Let X be the space of continuous complex-valued functions on $[-1, 1]$ and define an inner product on X by
$$(f, g) = \int_{-1}^{1} f(s)\overline{g(s)}\, ds\,.$$
Let $m(s)$ be a continuous function of absolute value 1, that is, $|m(s)| = 1$, $-1 \le s \le 1$. Define M to be multiplication by m:
$$(Mf)(s) = m(s)f(s)\,.$$
Show that M is unitary.

3. Fix an orthonormal basis of $\mathbb{C}^n$, and let S be the cyclic shift mapping $S(a_1, \dots, a_n) = (a_2, \dots, a_n, a_1)$.
   (a) Prove that S is unitary.
   (b) Find the characteristic and minimal polynomials, eigenvalues, and eigenvectors of S.
   (c) Find an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of S.

4. Consider the quadratic form $q(x) = 2x_1^2 + 6x_1x_2 + 2x_2^2$.
   (a) Write this as $q(x) = x^T Ax$, for some A.
   (b) Write $A = PDP^T$, where D is a diagonal matrix and P is orthogonal with determinant 1.
   (c) Change variables by letting $z = P^T x$. Sketch the level curve $q(x) = 1$ in both the z_1z_2-plane and in the x_1x_2-plane.

5. Let $N\colon X \to X$ be a linear map of a finite-dimensional complex inner product space. Prove that the following are equivalent:
   (i) N is normal (that is, $NN^* = N^*N$).
   (ii) N is unitarily similar to a diagonal matrix (i.e., $N = UDU^*$).
   (iii) Every eigenvector of N is an eigenvector of N^*.